PROGRESSIVE TRIGONOMETRY

MACMILLAN AND CO., LIMITED
LONDON · BOMBAY · CALCUTTA · MADRAS
MELBOURNE

THE MACMILLAN COMPANY
NEW YORK · BOSTON · CHICAGO
DALLAS · SAN FRANCISCO

THE MACMILLAN CO. OF CANADA, LTD.
TORONTO

PROGRESSIVE TRIGONOMETRY

PART I.

NUMERICAL TRIGONOMETRY AND MENSURATION

BY

FREDERICK G. W. BROWN

M.Sc. Lond., F.C.P.

SENIOR MATHEMATICAL MASTER, WEST HAM SECONDARY SCHOOL
AUTHOR OF "HIGHER MATHEMATICS FOR STUDENTS OF ENGINEERING AND SCIENCE"

MACMILLAN AND CO., LIMITED
ST. MARTIN'S STREET, LONDON
1928

PRINTED IN GREAT BRITAIN

PREFACE

The teaching of elementary mathematics, and particularly of geometry and trigonometry, has in recent years been made much more interesting and intelligible to beginners by the introduction of constructive work and measurement. By means of numerical trigonometry, which was advocated in 1911 by the Mathematical Association, not only has much of the drudgery involved in the older treatment of trigonometry been removed and the pupil's interest thereby stimulated, but it has been possible to begin the subject at a much earlier stage than was formerly the case. A firmer foundation is thus laid upon which a more serious treatment of the subject may be based.

It is pointed out in the Mathematical Association Report on the Teaching of Geometry (1925), that "The early work in trigonometry takes its place naturally in the course of elementary geometry," but as the treatment is at first mainly numerical, mensuration must be called upon to supply the concrete application of both geometrical and trigonometrical truths.

Hitherto mensuration has been usually included in text-books of arithmetic, and, as a consequence, its treatment has been somewhat restricted by the lack of trigonometrical application. The modern tendency, however, is to treat numerical trigonometry and mensuration together so that the scope of the latter subject may be considerably enlarged.

This book is therefore an attempt, founded upon experience, to combine mensuration with numerical trigonometry in a coherent scheme. The arrangement, however, is such that both mensuration and numerical trigonometry may, if preferred, be taken independently.

Although the character of the work is elementary, it covers practically all the syllabuses of school examining bodies, whether either mensuration or numerical trigonometry is required—or both. Another volume is in preparation which will deal with the trigonometry required to the end of a school course.

Fundamental rules are established in a series of numbered problems, and four-figure tables are used from the beginning, it being assumed that the reader has access to a handy collection of tables, such for example as Castle's *Logarithmic and Other Tables for Schools* (Macmillan, 6d.).

A large number of exercises has been provided: some quite straightforward and designed solely for drill purposes whereby accuracy and method may be developed; others, of a more practical nature, to illustrate the manifold applications of the subject—for herein lies the chief educational value of mensuration and trigonometry.

A few exercises have been taken from recent examination papers, and for permission to use these, thanks are gladly expressed to the Cambridge Local Examination Syndicate, the Central Welsh Board, the College of Preceptors, the Joint Matriculation Board of the Northern Universities, the National Union of Teachers, the Oxford Local Examinations Delegacy, the Union of Lancashire and Cheshire Institutes, and the Universities of Durham and London.

The author is specially indebted to Sir Richard Gregory for his constant help and wise counsel at every stage in the production of the book.

Thanks are due also to Mr. A. J. V. Gale, M.A., Mr. E. R. Shearmur, B.Sc., and Mr. G. D. Tyler, B.A., B.Sc., for kindly revising the proof sheets; to Mr. E. A. Branch for preparing many of the diagrams; and to the printers for the excellence of their work.

Answers have been supplied and carefully checked, but it is most probable that a few errors have escaped final detection. The author would therefore be glad to know of any corrections that readers may discover to be necessary.

F. G. W. BROWN.

GOODMAYES, ESSEX.

CONTENTS

The following abbreviations are used to indicate those exercises taken from examination papers :

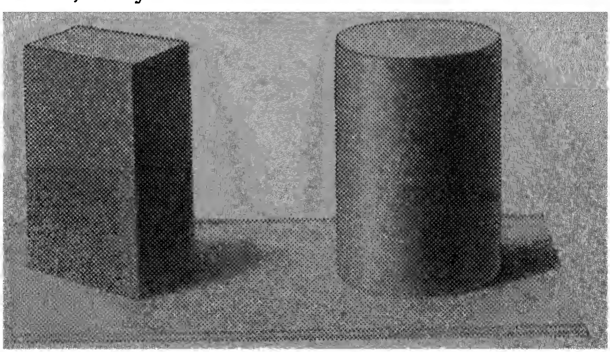
C.S.	Cambridge Senior.	
C.W.B.	Central Welsh Board.	
C.P.	College of Preceptors.	
J.M.B.	Joint Matriculation Board.	
N.U.T.	National Union of Teachers.	
O.J.	Oxford Junior.	
O.S.	Oxford Senior.	
U.L.C.I.	Union of Lancashire and Cheshire Institutes.	
D.S.	School Examination, University of Durham.	
G.S.	General School Examination,	University of London.
L.M.	Matriculation Examination,	University of London.
H.S.	Higher School Examination,	University of London.

CHAPTER I

MEASUREMENT OF LENGTH. APPROXIMATE CALCULATIONS

1. Introduction. The name "Mensuration" is derived from the Latin word *mensurare*, meaning to measure; hence the subject deals especially with measurement. We may define it as that part of Geometry which gives the rules for finding the lengths of lines, the areas of surfaces and the volumes of solids.

2. Definitions. Observe the two solid bodies depicted in Fig. 1. Each of them has length, breadth and thickness or height, so that we can measure them in three directions. This is true of all solid bodies, and the measurement in each direction is called a **dimension**: hence, every solid has **three** dimensions.



(a) FIG. 1. (b)

We see also that each solid has a boundary, which marks off the space occupied by the body from the surrounding space.

This boundary is called its total surface, and we notice that the total surface of the solid (*a*) consists of six flat portions, whilst that of the solid (*b*) consists of two flat ends and one curved portion. Each portion is, indeed, a surface in itself, and the flat faces are called plane surfaces to distinguish them from the curved surfaces.

As we are considering only the bounding faces of a solid, it will be easy to understand that a surface has no thickness. It does, however, possess length and breadth, so that a surface has **two** dimensions.

Lastly, we notice that each surface is bounded by lines, some straight as in solid (*a*), and some curved as in solid (*b*). Lines have no breadth or thickness, but length only: hence a line has only **one** dimension.

3. British Measurement of Length. All measurement really consists of comparing an unknown quantity with a standard quantity of the same kind: thus we might say that the length of this page is about one and a half pencil lengths, but this fact would be of little value unless everyone used a measuring pencil of the same length. It will, therefore, be clear that some standard length must be chosen. In Great Britain the unit adopted is called the **Imperial Yard.** This is defined by Act of Parliament as the distance at a temperature of 62° F. between two very fine lines, each engraved on a golden plug sunk at the ends of a bar of bronze. This bar is kept at the Standards Office in London, and copies of it are placed in the Houses of Parliament, the Royal Mint, the Royal Society's Office and the Royal Observatory, Greenwich.

British standards are defined by the Weights and Measures Act, passed in 1878. It is sometimes stated that Edward I. commanded that the length of his arm should define the yard, but this is not correct, though legislation in his reign established the standard.

The Imperial Yard, which has remained practically unchanged since that time, is divided into three feet, and each foot into twelve inches. The British table of linear measure is as follows:

12 inches	$=1$ foot
3 feet	$=1$ yard
$5\frac{1}{2}$ yards	$=1$ pole
40 poles	$=1$ furlong
8 furlongs	$=1$ mile.

A mile is thus equal to $8 \times 40 \times 5\frac{1}{2} = 1760$ yards, and is sometimes called a **statute mile** to distinguish it from a **geographical** or **nautical mile,** which is equal to 6080 feet, or 1·15 statute miles.

A speed of one nautical mile per hour is called a **knot,** and this is the unit used to express the rate at which a ship travels.

In sea soundings, etc., the depth of water is measured in **fathoms,** a fathom being 6 feet.

In the application of mensuration to land-surveying, distances are measured in links and chains, from the fact that a measuring chain 66 ft. long, containing 100 links, is used. This chain was devised about 1619 by Edmund Gunter, an English mathematician, and is therefore called **Gunter's chain.**

4. Metric Measurement of Length. The Metric or decimal system of measurement was originally intended to depend upon a natural unit of length, viz. one ten-millionth part of the distance from the North Pole to the Equator, measured along a meridian passing through Paris. The actual standard, however, is the distance at the melting-point of ice between the centres of two lines engraved upon a platinum bar kept at Paris. This distance is called the **International Metre.**

The Metric table of linear measure is as follows:

10 milli-metres (mm.)	$=1$ centi-metre (cm.)
10 centi-metres	$=1$ deci-metre (dm.)
10 deci-metres	$=1$ metre (m.)
10 metres	$=1$ deka-metre (Dm.)
10 deka-metres	$=1$ hecto-metre (Hm.)
10 hecto-metres	$=1$ kilo-metre (Km.)

5. Relationship between the Two Systems. If we measure the lengths of a few common objects in both Metric and British units, the relationship between the two systems may easily be found.

The following example, in which actual measurements are given, will shew how this may be done.

Ex. 1. *The lengths of six objects were measured in inches and centimetres with the following results :*

Inches - -	4·5	5·3	6·3	8·1	9·2	11·9
Centimetres -	11·5	13·5	16·1	20·5	23·3	30·1

Find the number of centimetres equivalent to an inch (i) *from each pair of measurements, and* (ii) *as the average of these six results.*

Taking the first pair of readings, since the same length is represented by 4·5 in. and 11·5 cm.,

$\therefore$ Number of cm. equivalent to one inch $= 11{\cdot}5 \div 4{\cdot}5$

$$= \frac{11{\cdot}5}{4{\cdot}5} = \frac{23}{9} = 2{\cdot}555\ldots = \mathbf{2{\cdot}56} \text{ to two places,}$$

since 2·56 is nearer 2·555... than 2·55.

Similarly, from the second pair, the required number of cm. is $13{\cdot}5 \div 5{\cdot}3 = 135 \div 53$. This will not cancel down any further; hence, by long division, $135 \div 53 = 2{\cdot}547$ to three places.

$\therefore$ To two places of decimals $13{\cdot}5 \div 5{\cdot}3 = \mathbf{2{\cdot}55}$.

In the same way:

$$16{\cdot}1 \div 6{\cdot}3 = 161 \div 63 = 2{\cdot}555\ldots = \mathbf{2{\cdot}56}.$$
$$20{\cdot}5 \div 8{\cdot}1 = 205 \div 81 = 2{\cdot}530\ldots = \mathbf{2{\cdot}53}.$$
$$23{\cdot}3 \div 9{\cdot}2 = 233 \div 92 = 2{\cdot}532\ldots = \mathbf{2{\cdot}53}.$$
$$30{\cdot}1 \div 11{\cdot}9 = 301 \div 119 = 2{\cdot}529\ldots = \mathbf{2{\cdot}53}.$$

The slight variation in these six results is due to the fact that the measurements are not absolutely exact. As they have been made to the nearest tenth of each unit, some may be slightly too great and some slightly too small. For practical purposes, however, the measurements are sufficiently true, and in order to compensate for the small errors, an average value is taken; thus from the six results, the average number of centimetres equivalent to an inch is

$$\tfrac{1}{6}(2{\cdot}56 + 2{\cdot}55 + 2{\cdot}56 + 2{\cdot}53 + 2{\cdot}53 + 2{\cdot}53)$$
$$= \tfrac{1}{6}(15{\cdot}26) = 2{\cdot}54.$$

Hence, **the average number of centimetres equivalent to one inch is 2·54.**

6. Graphical Representation. The relationship between the Metric and British units of length may very conveniently be shewn by means of a graph.

Ex. 2. *Construct a centimetre-inch graph, and shew how it may be used.*

Take a piece of squared paper ruled in tenths of an inch squares. Choose a scale of 5 squares to 2 cm. vertically and 5 squares to an inch horizontally. Referring to the table of measurements given in Ex. 1 (p. 4), take a point A (Fig. 2) 4·5 in. from OY and

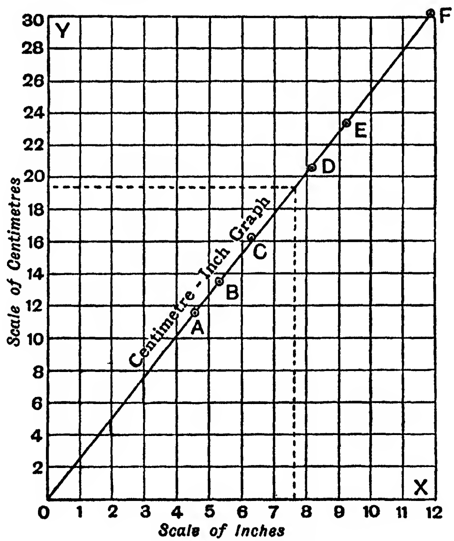


FIG. 2.—Centimetre-Inch Graph.

11·5 cm. from OX. This point then respresents the first pair of measurements. Similarly, mark points B, C, D, E, F representing the remaining five pairs of measurements. Notice that these points lie very nearly in a straight line. Draw from O the straight line passing through as many of the points as possible, as shewn in Fig. 2.

This is called a **centimetre-inch graph.** It is very useful, for it shews at a glance the number of inches in a given number of centimetres, as well as the number of centimetres in a given

number of inches. Thus, for example, suppose we wish to know the number of cm. equivalent to 7·6 in. Looking along OX for 7·6 in. we follow the vertical line there up to the graph, and then the corresponding line horizontal from the graph to the scale OY. This meets OY at 19·3 cm., so that 19·3 cm. are equivalent to 7·6 in. Similarly, we find the equivalent of 24 cm. is 9·4 in. From these results it is easy to see that the equivalents of 0·76 or 76 in. are 1·93 or 193 cm. respectively. Likewise, in the case of other measurements the scales may be mentally extended or reduced by multiplication or division by ten.

7. Approximate Calculation. We have seen from Ex. 1 (p. 4) that the division of actual measurements is rarely exact, and this is true of all practical calculations. Indeed, no measurements can be made with absolute accuracy, and, as a consequence, when using measured values in calculations, we must not use more figures than are necessary to give reliable results. To do this we use what are called **Contracted Methods,** and the following examples will shew how these may be employed.

Ex. 3. *Taking one inch to be equivalent to* 2·54 *cm., find, by contracted methods, the number of*

(i) *kilometres in a mile* *and* (ii) *inches in a metre,*

each to two places of decimals.

(i) 1 mile = 1760 × 36 in. = 63360 in. = 63360 × 2·54 cm.
Now 1 kilometre = 100,000 cm.

$$\therefore\ 1 \text{ mile} = 63360 \times 2{\cdot}54 \div 100{,}000 \text{ km.}$$
$$= 0{\cdot}6336 \times 2{\cdot}54 \text{ km.}$$

This product must be calculated correctly to two places of decimals. We therefore proceed as follows:

0·63	36	
2·54		
1·26	72	
·31	65	·
2	52	· ·
1·59	89	· ·

Rough check:
0·6 × 2·5 = 1·5.

In order that the decimal points should all lie under one another, the multiplier must have one digit in front of the decimal point. It is often necessary to move the point in both multiplicand

nd multiplier in order to satisfy this rule. In the above case, no revious adjustment is needed as the multiplier, 2·54, has already ne digit in front of the point.

Since the result is required correctly to two places, a vertical ine is drawn two places after the decimal, and then to ensure the ccuracy of the second figure, another vertical line is drawn after wo places further to the right. Beyond this line no figures must ppear, but their places indicated by dots.

The digits in the final product between the two vertical lines re not required; they have only been found to ensure the ccuracy of the second place. Hence if either of them is less han 5, it may be neglected. If, however, either of them is 5 r greater than 5, then it can only be omitted by increasing the ext digit on the left by 1; thus the above product becomes 1·60, hich is correct to two places.

Hence 1 mile is equivalent to 1·6 kilometres,

.e. **5 miles are equivalent to 8 kilometres.**

(ii) 1 metre = 100 cm. = 100 ÷ 2·54 inches.

To perform this division approximately to two places of ecimals, we must first adjust the decimal points so that the ivisor has one digit in front of the point, just as in multiplication.

$$
\begin{array}{r|l|}
39{\cdot}37 & 0 \\
2{\cdot}54\,)\,100{\cdot}00 & 00 \\
76{\cdot}2 & \\
23{\cdot}80 & \\
22{\cdot}86 & \\
{\cdot}94 & 0 \\
76 & 2 \\
17 & 80 \\
17 & 78 \\
 & 2
\end{array}
$$

Two vertical lines should then be drawn, one immediately after the number of places required, the other, two places further to the right; no figures must be placed to the right of this line.

The quotient must always be placed above the dividend, so that all the points lie in the same vertical line.

The actual division appears thus on the left.

A rough check is 100 ÷ 2·5 = 40.

Hence the required quotient correct to two places is 39·37.

∴ **One metre is equivalent to 39·37 inches.**

This is approximately equal to 1·09 yards or 3 ft. $3\frac{1}{3}$ in.

Hence, the following table of equivalents between the two ystems; the sign ≡ meaning 'is or are equivalent to.'

1 inch ≡ 2·54 centimetres.

5 miles ≡ 8 kilometres.

1 metre ≡ 39·37 inches or 1·09 yards.

8. Further Examples in Approximation. In order to illustrate more fully the brief rules already given for contracted methods of calculation, a few more difficult examples are worked out below.

Ex. 4. *On a drum there are* 63·8 *turns of wire, and in each turn there are* $82\frac{1}{16}$ *inches of wire ; find the total length of wire on the drum correctly to the nearest tenth of an inch.*

$82\frac{1}{16}$ inches expressed as a decimal is 82·0625 inches, hence the required length $=82{\cdot}0625\times63{\cdot}8$.

Since there are two figures in front of the decimal point in the multiplier, we must move the point one place to the left, *i.e.* divide 63·8 by 10, and to preserve the value of the product, the multiplicand must be multiplied by 10, *i.e.* the decimal point must be moved one place to the right.

Hence $82{\cdot}0625\times63{\cdot}8=820{\cdot}625\times6{\cdot}38$, and the multiplication may now be carried out as follows :

820·6	25
6·3	8
4923·7	50
246·1	86 .
65·6	48 . .
5235·5	84 . .

Rough check :
$82\times64=5248$.

Hence the total length of wire, correct to the nearest tenth of an inch, is **5235·6** inches.

Ex. 5. 983·9 *metres of cloth have to be cut into strips each* 56·84 *metres in length. Find the number of strips to the nearest tenth.*

The required number of strips is $983{\cdot}9\div56{\cdot}84$, or, putting the divisor in standard form,

$$98{\cdot}39\div5{\cdot}684.$$

	17·3	1
5·6$\cancel{8}\cancel{4}$)	98·3	90
	56·8	4
	41·5	50
	39·7	88
	1·7	62
	1·7	04
		58
		56
		2

Rough check :

$$99\div6=16{\cdot}5.$$

It should be observed that when there are no further digits or cyphers to bring down from the dividend, the division is continued by crossing off the right-hand digit of the divisor, as indicated on the left.

Hence the required number of strips is **17·3**.

Ex. 6. *Calculate, correctly to two places of decimals, the value of* 2·17732 × 9·08813 ÷ 1·82641.

Here there are two operations to be performed, and in such cases each should be worked to one place beyond the number required. The working appears thus:

2·177	32
9·088	13
19·595	88
·174	16
17	36
	21
	6
19·787	67

= 19·788,

10·83	4
1·82641)19·78	80
18·26	41
1·52	39
1·46	08
6	31
5	46
	85
	72
	13

Hence the required result is **10·83**.

A rough check may readily be made here, for the given expression may be replaced by the following:

$$2{\cdot}2 \times 9 \div 2 = 9{\cdot}9,$$

which agrees roughly with the actual value obtained.

It should be remembered that a rough check should always be made, as it indicates whether the value calculated by the contracted method is of the right order of magnitude. If, for instance, the value of the above calculation had been 108·3 or 1·083, then the roughly estimated result, 9·9, would shew at once that neither of these values could be correct, because there is a disagreement in the orders of magnitude.

EXERCISES 1.

Calculate correctly to two places of decimals:

1. 834·65 × 462·35.
2. 971·6834 × 39·24.
3. 3·1416 × 3·1416.
4. 40·087 × 0·049.
5. 32·578 × 57·913.
6. 72·5318 ÷ 9·57.
7. 8·9563 ÷ 14·86.
8. 1 ÷ 3·1416.
9. 0·579 ÷ 32·18.
10. 39·8126 ÷ 51·73.

Calculate correctly to four significant figures:

11. $\dfrac{4{\cdot}283 \times 7{\cdot}183}{6{\cdot}195}$.

12. $\dfrac{42{\cdot}67 \times 0{\cdot}8632}{148{\cdot}6}$.

13. $\dfrac{86{\cdot}92}{5{\cdot}873 \times 0{\cdot}762}$.

14. $\dfrac{342{\cdot}7}{48{\cdot}36 \times 5{\cdot}374}$.

15. $\dfrac{9876 \times 1235}{654{\cdot}3 \times 567{\cdot}9}$.

16. $\dfrac{14{\cdot}62 \times 3{\cdot}14}{18{\cdot}73 \times 0{\cdot}4362}$.

17. Divide 3·758 by 72·358, and verify the result by multiplication, getting your results accurate to the fifth place of significant figures, and using as few figures as are necessary for this degree of approximation. (C.S.)

18. Multiply 1·183 by 2·145 and divide the result by 0·0845. (C.S.)

19. Find the value of $13{\cdot}86 \times 17{\cdot}05 \div 8{\cdot}93$ correct to two places of decimals. (C.S.)

20. Find the value of $\dfrac{3{\cdot}842 \times 0{\cdot}0025}{0{\cdot}0138}$ correct to three places of decimals. (C.S.)

21. If $\dfrac{1}{u} + \dfrac{1}{v} = \dfrac{1}{f}$, find the value of f, correct to three significant figures, when $u = 2{\cdot}36$, $v = 8{\cdot}64$. (L.M.)

22. Taking one inch to be equivalent to 2·54 centimetres, find the number of metres, correct to two places of decimals, equivalent to a chain. Use this value to find the number of chains in 135·81 metres.

23. Find the number of nautical miles in 19 statute miles, and the number of statute miles in 19·8 nautical miles.

24. The distance between two stations on a railway is 9420 metres. Taking 5 miles as the equivalent of 8 kilometres, find the number of chains in this distance.

25. Find the number of complete kilometres in 3040 miles, taking a centimetre to be 0·3937 inch. (C.S.)

26. Subtract 0·57 kilometres from 3780 metres and give the answer in yards, taking 8 kilometres as being equivalent to 5 miles. (C.S.)

27. A surveyor's chain should measure 22 yards, but has stretched 4 inches. If he measures a given length with this chain, find to three decimal places what percentage of his measured length is error. (L.M.)

28. How many pieces of wire, each 4·13 inches long, can be cut from a rod whose length is 8·24 feet, and what will be the length of the piece left over? (C.S.)

29. Having given that a metre is 39·3708 inches, shew that the difference between 35 yards and 32 metres is less than a centimetre. (C.S.)

30. How many pieces, each 3 metres 5 centimetres long, can be cut off a rope 35 metres long? What is the length of the remainder? (C.S.)

31. A straight rod is 3 ft. 5 in. long. How many times must it be laid along a straight border to measure a length of 75 metres, taking 1 m. $=3{\cdot}28$ ft.?

32. A strip of steel is $9\frac{3}{4}$ inches long. How many pieces, each of length $\frac{3}{4}$ inch, may be cut from it if $\frac{1}{32}$ inch is wasted for each piece sawn off? What is the length of the piece remaining? (U.L.C.I.)

33. A bar of iron 19 feet 1 inch long is to be sawn into 13 pieces of equal length. Assuming the saw cut to be $\frac{1}{8}$ inch wide, find the length of each part, and the percentage waste.

34. The following is a rule for turning feet into metres:

(i) Multiply the number of feet by 3 and divide by 10.

(ii) Multiply it by 5 and divide by 1000.

(iii) Multiply it by 2 and divide by 10,000.

Add the first two of these results and subtract the third.

Prove that the rule is correct to about one part in a million, being given that one foot is equal to 0·30479972 metre.

The height of Mount Everest is stated to be 29,141 ft., find it in metres to the nearest metre. (L.S.)

35. One inch of steel expands $\frac{1}{150000}$ inch for 1° F. rise in temperature; what will be the increase in length of a steel bar 30 ft. long for 255° F. rise in temperature? Give the answer in inches. (U.L.C.I.)

36. A French map is drawn on a scale of one centimetre to one kilometre. Find the distance in English statute miles between two places which are 9·9 inches apart on the map.

37. How many miles are represented by one inch on a map which shews five centimetres between two places that are really 15·84 kilometres apart?

38. If the distance of the surface of the sun from the earth is 92,400,000 miles, and the velocity of light is 300,000 kilometres per second, and one mile $=1{\cdot}61$ km., find the time light takes to travel from the sun to the earth in minutes and seconds. (L.M.)

39. One inch of steel increases 0·0000105 in. in length for 1° C. rise in temperature. What will be the increase in the length of a steel rail 30 ft. long for a rise of 42° C.?

40. A straight bar is made by joining a bar of brass 16 ft. 8 in. long end to end to a bar of iron 50 ft. long. When heated through 120° C. the total increase in the length of the compound bar is 1·248 in. If one inch of brass increases 0·0000193 in. in length for a rise of 1° C., find how much one inch of iron increases in length for a rise of 1° C.

CHAPTER II

ANGLES AND THEIR MEASUREMENT

9. Definition of an Angle. Suppose we take two straight but very thin rods OX, OM (Fig. 3), pivoted at O, and holding OM

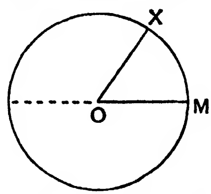


FIG. 3.—Definition of an angle.

fixed, turn OX in an anticlockwise direction, then OX describes positive angles which are measured by the amount of turning MX. The angle described by a complete revolution of OX is divided into 360 equal parts called **degrees**, and a quarter of a revolution is called a **right angle**. Thus

a complete revolution = 360 degrees = 4 right angles,

so that each right angle contains **90 degrees**.

Each degree is subdivided into 60 **minutes**, and each minute into 60 **seconds**. Degrees, minutes and seconds are briefly denoted by $^\circ$, $'$, $''$ respectively, so that

$$60'' = 1'; \; 60' = 1^\circ; \; 90^\circ = 1 \text{ right angle.}$$

The instrument used for measuring angles is called a **Protractor**, and usually consists of a semicircular plate divided into degrees round its circumference.

10. Angles measured by Ratios. The following exercise will shew that there are other methods of measuring angles when we have a suitable set of tables.

Ex. 7. *Shew how angles may be determined by measuring the ratios of the sides of a right-angled triangle.*

Take a piece of squared paper accurately ruled in millimetre squares, and by means of a good protractor draw an angle XOQ

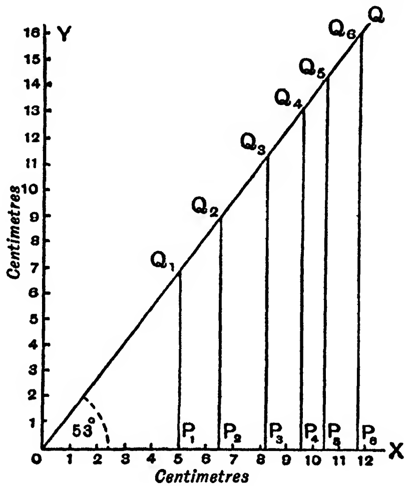


FIG. 4.

containing 53° (Fig. 4). Along OX mark a number of points P_1, P_2, P_3, ... , and at each of these points draw lines perpendicular to OX meeting OQ in Q_1, Q_2, Q_3, ... Read off the lengths of OP_1, P_1Q_1; OP_2, P_2Q_2; OP_3, P_3Q_3; ... , and then calculate the values of $P_1Q_1 \div OP_1$; $P_2Q_2 \div OP_2$; $P_3Q_3 \div OP_3$; ... to two places of decimals.

Thus, for Fig. 4, we have the following results:

$OP_1 =$ 5·0 cm.	$P_1Q_1 =$ 6·7 cm.	$P_1Q_1 \div OP_1 = 1{\cdot}34$
$OP_2 =$ 6·5 „	$P_2Q_2 =$ 8·7 „	$P_2Q_2 \div OP_2 = 1{\cdot}34$
$OP_3 =$ 8·2 „	$P_3Q_3 =$ 10·9 „	$P_3Q_3 \div OP_3 = 1{\cdot}33$
$OP_4 =$ 9·6 „	$P_4Q_4 =$ 12·7 „	$P_4Q_4 \div OP_4 = 1{\cdot}32$
$OP_5 =$ 10·4 „	$P_5Q_5 =$ 13·8 „	$P_5Q_5 \div OP_5 = 1{\cdot}33$
$OP_6 =$ 11·7 „	$P_6Q_6 =$ 15·4 „	$P_6Q_6 \div OP_6 = 1{\cdot}32$

These results shew that the ratios obtained are very nearly the same. Indeed, if we could make perfectly accurate measurements

the ratios would not vary at all. It is evident, therefore, that for an angle of 53° each of the bases OP_1, OP_2, ... , OP_6 is a constant fraction of its corresponding perpendicular P_1Q_1, P_2Q_2, ... , P_6Q_6.

If we repeat the process by working through Exercises 2A, it will be found that the ratios are the same for the same angle, but are different for different angles; thus for 36° the ratio is 0·73, for 42° it is 0·90, for 70° it is 2·75, the ratio increasing as the angle increases. If, therefore, we know the ratios for all the angles from 0° to 90°, we shall have another method by which the size of an angle may be estimated to a fair degree of accuracy.

EXERCISES 2A.

Repeat Ex. 7 in the case of each of the following angles, making measurements in centimetres for Nos. 1-6, and in inches for Nos. 7-12.

1. 10°.	2. 18°.	3. 23°.
4. 28°.	5. 33°.	6. 40°.
7. 45°.	8. 56°.	9. 60°.
10. 69°.	11. 80°.	12. 85·6°.

11. The Tangent of an Acute Angle. An acute angle is one which is less than a right angle. If BAC (Fig. 5) be such an angle, it will be seen from the results already obtained, that if from any point C in AC we draw CB perpendicular to AB, then the ratio BC÷AB, or, as we shall often write it, BC/AB, will depend entirely upon the size of the angle A, *i.e.* ∠BAC. We shall call this ratio the **tangent of the angle A**, and shall write it in the usual short form, **tan A,** so that

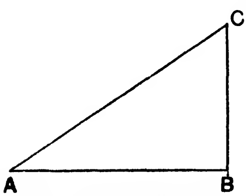


FIG. 5.—Tangent of an acute angle.

$$\tan A = BC/AB.$$

In considering the angle A, since ABC is a right-angled triangle, we shall call the arms AB, AC, the **base** and **hypotenuse** respectively, whilst BC will be termed the **perpendicular.**

Hence we see that tan A = (perpendicular BC)/(base AB), or omitting the letters, the **tangent of an acute angle = ratio of the per-**

pendicular to the base, where the base is one of the arms of the angle and the perpendicular is a line drawn at right angles to the base from any point on the other arm of the angle.

The results of Ex. 7, p. 13, may now be written down briefly in the form $\tan 53° = 1{\cdot}33$, this being the average value of the six ratios calculated. Tables have been constructed, giving the tangent of every acute angle correctly to four places of decimals and the use of them enables many problems to be solved in quite a simple way.

Ex. 8. *Construct an angle whose tangent is* 1·6.

Suppose the angle A in Fig. 5 (p. 14) to be such that its tangent is 1·6; then we should have $\tan A = BC/AB = 1{\cdot}6$, so that

$$BC = 1{\cdot}6 \times AB.$$

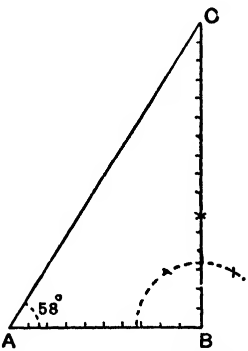


FIG. 6.—Construction of an angle whose tangent is given.

If, therefore, the length of AB be known, we can easily find the corresponding length of BC. This leads to a simple construction.

Draw a straight line AB (Fig. 6) of any convenient length. Suppose we make it 10 cm. long; then

$$BC = 1{\cdot}6 \times AB = 1{\cdot}6 \times 10 = 16 \text{ cm.}$$

Hence at B, erect a perpendicular to AB and on it mark off $BC = 16$ cm. Join CA, then $\angle BAC$ is the required angle; for

$$\tan A = BC/AB = 16/10 = 1{\cdot}6.$$

On measuring the angle carefully with a protractor, we find that its size is approximately 58°. Hence $\tan 58° = 1{\cdot}6$.

Ex. 9. *In a triangle* ABC, *the angle* C *is a right angle. Find the value of tan* A × *tan* B (i) *when* A = 67° *and* (ii) *when* B = 44°, *having given that*

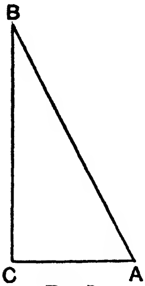


FIG. 7.

$$tan\ 23° = 0{\cdot}4245, \qquad tan\ 44° = 0{\cdot}9657.$$
$$tan\ 46° = 1{\cdot}0355, \quad and \quad tan\ 67° = 2{\cdot}3559.$$

What do you notice about the results ?

The triangle is shewn in Fig. 7, and we know from geometry that the sum of the three angles of every triangle is 180°.

Hence, since

$$C = 90°,\ A + B = 90°.$$

$$\therefore \text{ when } A = 67°,\ B = 90° - 67° = 23°,$$
$$B = 44°,\ A = 90° - 44° = 46°.$$

To find the product of tan A and tan B, we must therefore multiply (i) 0·4245 by 2·3559, and (ii) 0·9657 by 1·0355.

Working by contracted methods, we have

$$\text{(i)}\quad \begin{array}{r|l|l}
0{\cdot}424 & 5 & \\
2{\cdot}355 & 9 & \\
\hline
0{\cdot}849 & 0 & \\
{\cdot}127 & 35 & \\
21 & 20 & \cdot \\
2 & 10 & \cdot\ \cdot \\
 & 36 & \cdot\ \cdot\ \cdot \\
\hline
1{\cdot}000 & 01 &
\end{array}
\qquad\qquad
\text{(ii)}\quad \begin{array}{r|l|l}
0{\cdot}965 & 7 & \\
1{\cdot}035 & 5 & \\
\hline
0{\cdot}965 & 7 & \\
28 & 95 & \cdot \\
4 & 80 & \cdot\ \cdot \\
 & 45 & \cdot\ \cdot\ \cdot \\
\hline
0{\cdot}999 & 90 &
\end{array}$$

It is evident that the product in each case is practically 1.

A little consideration of Fig. 7 will shew us that this is always the case when A + B = 90° : for tan A = BC/CA, and for ∠ B, BC is one arm and therefore the *base* whilst CA is the *perpendicular* so that tan B = CA/BC.

Hence, in all cases when A + B = 90°,

$$\mathbf{tan\ A\ .\ tan\ B} = \frac{BC}{CA} \cdot \frac{CA}{BC} = \mathbf{1}.$$

12. Angles of Elevation and Depression. Suppose an observer stands at G (Fig. 8) and, facing a vertical wall KH, looks at the top H ; then the line joining his eye E with the top H makes an angle AEH with the horizontal line EA, which is called the **angle**

of **elevation** of H. Since tan AEH = AH/EA, we can find the height AH if the angle AEH and the distance EA or GK be known. Also KA = GE = height of the observer's eye above the ground GK, and if this be measured, the total height of the wall is KH = KA + AH.

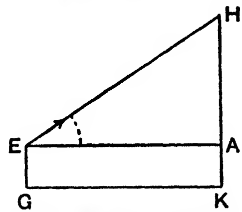


FIG. 8.—Angle of elevation.

Suppose, however, that the observer were standing above the object observed, as when a person stands on the edge of a cliff P (Fig. 9) and looks at a distant ship B. The angle BPQ made by

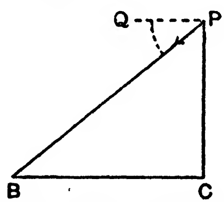


FIG. 9.—Angle of depression.

the horizontal line PQ with the line PB joining the ship and his eye is called the **angle of depression** of B. It will be clear from the figure that ∠BPQ = ∠CBP, so that tan BPQ = tan CBP = PC/BC. Hence, when the angle and one of the distances PC or BC are known, the other distance can easily be found.

Angles of elevation and depression are measured by an instrument called a **theodolite**, which consists of a telescope made to revolve in a vertical plane over a graduated circle whose centre is at the pivot of the telescope. The whole is mounted on a horizontal table supported on three legs. For ordinary purposes the telescope may be replaced by a simple sighting device, and a very serviceable and reliable instrument may be constructed in

a school woodwork shop. Fig. 10 shews such an instrument. Note the spirit level for the horizontal adjustment of the table.

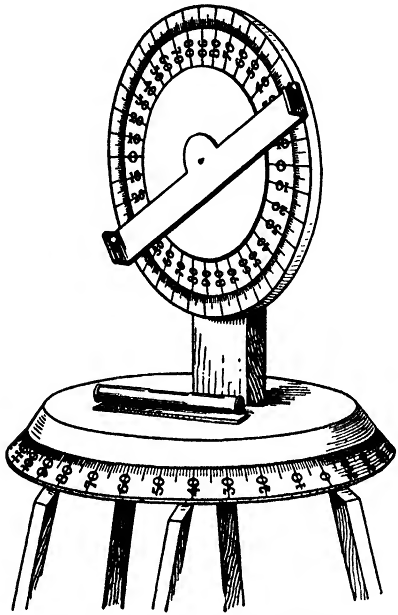


FIG. 10.—A school-made theodolite.

Ex. 10. *A boy observes that the angle of elevation of the top of the bell-tower of his school is* 61° *at a horizontal distance of* 65 *feet from the foot of the tower. If his measuring instrument be* 4 *ft. from the ground, find the height of the tower.*

Referring to Fig. 8, let AH represent the part of the tower above the measuring instrument E, then $\angle AEH = 61°$, and from the table of tangents, $\tan 61° = 1{\cdot}8040$.

But $\tan AEH = AH/EA$, and $EA = GK = 65$ ft.

$$\therefore\ 1{\cdot}8040 = AH/65\,;$$

$$\therefore\ AH = 1{\cdot}8040 \times 65 = 117{\cdot}26 \text{ ft.}$$

Also $$KA = GE = 4 \text{ ft.}$$

$$\therefore\ KH = KA + AH = 4 + 117{\cdot}26 = 121{\cdot}26 \text{ ft.}$$

$\therefore$ height of tower to the nearest tenth of a foot = **121·3** ft.

Ex. 11. *From the top of a church tower a man observes the bank of a river which he knows is approximately 34 yards from the foot of the tower. He finds that its angle of depression is 64°; find the height of the tower in feet.*

In Fig. 9, let CP represent the tower, and B the bank of the river; then $BC = 34$ yd. $= 102$ ft., and $\angle QPB = \angle CBP = 64°$.

From the table of tangents, $\tan 64° = 2{\cdot}0503$, and

$$PC/BC = \tan 64°, \text{ or } CP/102 = 2{\cdot}0503,$$

$$\therefore \; PC = 2{\cdot}0503 \times 102 = \mathbf{209{\cdot}1} \text{ ft.}$$

EXERCISES 2B.

1. Draw any acute angle POQ; take any point Q in OQ and draw QS perpendicular to OP meeting it in S. Take any point P in OP and draw PR perpendicular to OR meeting it in R. Measure OS, SQ, OR, RP, and calculate the values of the ratios SQ/OS, RP/OR. What do you deduce from the results? State what each ratio denotes.

Measure the angle POQ and verify your results from the tables.

2. Draw a straight line OP $4\frac{1}{4}$ inches long; mark off $OS = 3$ in., and at S draw $SQ = 1{\cdot}6$ in. at right angles to OS. Join OQ and produce it. From P draw PR perpendicular to OQ produced to meet it in R. Measure PR and OR, hence calculate the tangents of the angles OQS, OPR. What do the results shew? Measure the angles and verify the measurements from the tables.

3. Draw a right angle POQ having its arms OP, OQ any convenient length; join PQ. Draw OR perpendicular to PQ meeting it at R. Measure OR, RP, QR; calculate the values of the ratios QR/OR, and OR/RP. What do your results shew? Verify your inference by measuring the angles OPR, ROQ.

4. Draw a triangle POQ having $\angle OPQ = 90°$, $OP = 12$ cm., and $PQ = 3{\cdot}5$ cm. Calculate the tangents of the angles POQ and PQO, and find the relation between them.

Construct and measure the angles whose tangents are

5. 11·2. **6.** 13. **7.** 0·7. **8.** 0·74.

9. 0·36. **10.** 1·2. **11.** 0·4. **12.** 2·5.

13. Construct the angles whose tangents are $\frac{3}{7}$ and $\frac{2}{8}$; measure them and find their sum.

14. Construct the angles whose tangents are 0·4 and 2·5; measure them and find their sum.

15. Construct two angles whose tangents are 10·2 and 0·3 respectively; measure them and find their difference.

16. Write down from the tables the tangents of $18° 6'$, $19°$, and $35°$. Then find the angle whose tangent is equal to

$$\tan 18° 6' + \tan 19° + \tan 35°.$$

17. Write down the tangents of $10°$, $20°$, and $40°$, and find the angle whose tangent is $\tan 20° + 2 \tan 40° + 4 \tan 10°$.

18. Write down $\tan 84° 24'$, $\tan 72° 36'$, $\tan 85° 42'$; hence find the angle whose tangent is $\tan 84° 24' + \tan 72° 36' - \tan 85° 42'$.

19. Write down the tangents of $16° 42'$ and $67° 18°$; hence evaluate

$$(\tan 16° 42' + \tan 67° 18') \div (1 - \tan 16° 42' \times \tan 67° 18'),$$

and find the angle whose tangent is equal to this value. What do you notice about this angle and the sum of the given angles ?

20. Write down the tangents of $84° 54'$ and $21° 48'$; hence evaluate

$$(\tan 84° 54' - \tan 21° 48') \div (1 + \tan 84° 54' \times \tan 21° 48'),$$

and find the angle whose tangent is equal to this value. What do you observe about this angle and the difference between the given angles ?

21. If $\tan \theta = \dfrac{\tan \mathsf{A} + \tan \mathsf{B}}{1 - \tan \mathsf{A} \,.\, \tan \mathsf{B}}$, find θ when $\mathsf{A} = 26° 36'$ and $\mathsf{B} = 50° 12'$. What is the relation between θ and $\mathsf{A} + \mathsf{B}$?

22. Given that $\tan \theta = (\tan \mathsf{A} - \tan \mathsf{B})/(1 + \tan \mathsf{A} \,.\, \tan \mathsf{B})$, find θ when $\mathsf{A} = 84° 18'$ and $\mathsf{B} = 31° 48'$. What is the relation between θ and $\mathsf{A} - \mathsf{B}$?

23. In a triangle ABC, the angle C is a right angle, the side BC = 38·48 ft., and the side CA = 49·25 ft. Find the angles at A and C.

24. A wire attached to the top of a vertical post is fastened on the ground at a point 68 ft. from the foot of the post and makes an angle of $73°$ with the ground. Find the height of the post.

25. A ladder is placed against a vertical wall with its foot 5·4 ft. away from the wall. The inclination of the ladder to the ground is $75°$; how far up the wall does the top of the ladder reach ?

ABC is a triangle having a right angle at C ; if :

26. $\angle \mathsf{B} = 35°$, CA = 10·5 in., find BC.

27. $\angle \mathsf{A} = 60° 42'$, BC = 26·73 in., find CA.

28. $\angle \mathsf{B} = 69° 12'$, BC = 12 cm., find CA.

29. $\angle \mathsf{A} = 70°$, CA = 24 cm., find BC.

30. $\angle \mathsf{B} = 70° 6'$, BC = 16 in., find CA.

31. $\angle \mathsf{B} = 72° 48'$, CA = 64·61 ft., find BC.

32. $\angle \mathsf{A} = 49° 36'$, BC = 4·8 ft., find CA.

33. $\angle \mathsf{A} = 21° 48'$, CA = 14 in., find BC.

34.* ABC is an isosceles triangle whose base BC is 6·4 inches long, and whose height is 6·2 in. Find its angles.

35. The base BC of an isosceles triangle ABC is 8 in., and its height is 11·75 in., find its angles.

36. Each of the equal angles at the base of an isosceles triangle is 84° 24′, and its height is 12 ft. 9 in. Find the length of the base.

37. The base of an isosceles triangle is 2 ft. 1 in. long, and each of its equal angles is 54°; find its height.

38. An isosceles triangle has a vertical angle of 39° 36′; its base is 4·8 ft. long. Find its base angles and its height.

39. The height of an isosceles triangle is 10·4 cm., and its vertical angle is 27° 48′; find its base angles and the length of the base.

In a triangle ABC, the perpendicular from A to BC meets it in D; if

40. $BD=5$, $DC=4$, $AD=9{\cdot}02$; find $\angle B$ and $\angle C$.

41. $BD=14$, $BC=21$, $\angle B=55°\ 30'$; find AD and $\angle C$.

42. $\angle B=43°$, $\angle C=35°\ 18'$, $AD=37{\cdot}3$; find BD and DC.

43. $\angle B=24°$, $BD=11{\cdot}23$; find AD, then find $\angle C$ if $DC=5$.

44. $\angle C=16°\ 42'$, $AD=8{\cdot}1$, $BC=12{\cdot}1$; find DC and $\angle B$.

45. $35AD=14BD=10BC$; find $\angle A$, $\angle B$ and $\angle C$.

46. Two flag-poles of heights 43 ft. and 35 ft. respectively are erected on opposite sides of a road 37 ft. wide. Find the angle made with the horizontal by a line joining the tops of the poles. (N.U.T.)

47. In a triangle ABC, $\angle B=40°$, $\angle C=50°$, and the perpendicular from A on BC is 10 feet. Calculate the length of BC. (C.S.)

48. A statue 25 feet high stands on the top of a column 100 feet high. Calculate the angle of elevation of the top of the statue at a point 200 feet from the base of the column in a horizontal line. What angle does the statue subtend at this point? (D.S.)

49. An aeroplane when 2118 feet high passes vertically above another at an instant when the angles of elevation at the same observing point are 46° and 40° respectively. How many feet lower is the one than the other? (D.S.)

50. The top and bottom of a vertical flag-pole, fixed on the top of a tower, are observed from a point on the ground distant 125 ft. from the foot of the tower, and the angles of elevation are found to be 27° 9′ and 22° respectively. Find the height of the tower and the length of the pole.

51. At a point 25 feet distant from the wall of a house, the angles of elevation of the upper and lower edges of a window are $38° 40'$ and $27° 30'$ respectively. Calculate the height of the window. (C.S.)

52. A flag-staff, 10 feet long, stands on the top of a tower. To an observer, whose eye is on the same level as the foot of the tower and 50 yards from it, the angle of elevation of the top of the flag-staff is $42°$. Calculate the height of the tower, to the nearest foot, and the angle the flag-staff subtends at the observer's eye, to the nearest minute. (C.S.)

CHAPTER III

MEASUREMENT OF AREA. RECTILINEAL FIGURES

13. Rectilineal Figures. If a figure be drawn on a sheet of paper to represent one of the faces of the solid shewn in Fig. 1*a* (p. 1), we have a plane figure bounded by four straight sides. Now all plane figures which are bounded only by **straight lines,** no matter how many there may be, are called **rectilineal figures.**

When there are only **three sides,** the figure is a **triangle,** as in Fig. 11.

When there are **four sides,** the figure is called **a quadrilateral,** as in Fig. 12. A quadrilateral is known as

(i) a **rectangle,** if all its angles are right angles, as in Fig. 13.

(ii) a **parallelogram,** if both pairs of opposite sides are parallel, as in Fig. 14.

FIG. 11.—A triangle.

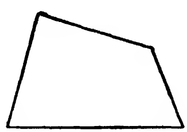


FIG. 12.—A quadrilateral.

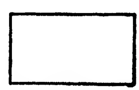


FIG. 13.—A rectangle.

FIG. 14.—A parallelogram.

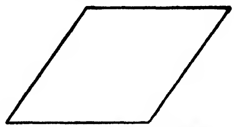


FIG. 15.—A rhombus.

FIG. 16.—A trapezium.

(iii) a **rhombus,** if both pairs of opposite sides are parallel and all its sides are equal, as in Fig. 15.

(iv) a **trapezium,** if one pair of opposite sides are parallel, as in Fig. 16.

Each of these figures, like all other plane figures, has two dimensions, *i.e.* length in two directions which we often call **length** and **breadth.** The measurements of these are always made in directions which are perpendicular to each other.

14. Area of Rectangle. The measure of the space enclosed by the boundaries of a plane figure is called its **area,** and this may be found from the measurements of the length and breadth of the figure.

PROBLEM 1. *To find the area of a rectangle.*

On a piece of paper ruled in $\frac{1}{10}$ inch squares, draw a rectangle 32 units long and 17 units broad, taking the side of a small square as a unit. How many squares are enclosed by the figure? Clearly the number is $32 \times 17 = 544$. Now the unit of area must be an area itself, and since our two measurements are perpendicular to each other in direction, the most convenient unit of area is the square.

When the side of the unit square is one inch, the area of the square is **1 square inch**; if the length of the side be one yard, then its area is **1 square yard,** and so on. The measure of an area is thus the number of unit squares contained in it. The area of our rectangle is therefore 544 unit squares, each of whose sides is $\frac{1}{10}$ inch.

Now the length of the rectangle is 32 units each $\frac{1}{10}$ inch long, so that really it is 3·2 in. Similarly, the actual breadth is 1·7 in. But each large square whose side is one inch, *i.e.* a square inch, contains 100 small squares, so that 544 small squares contain 5·44 sq. in. This, then, is the area of a rectangle 3·2 in. by 1·7 in.

If, however, we multiply 3·2 by 1·7 we obtain 5·44; hence the **area of a rectangle is measured by the product of the length and breadth,** provided always the length and breadth are measured in the same units.

Ex. 12. *Find the area of a rectangle whose sides are* 23·5 *in. and* 17·6 *in.*

Evidently, from the above rule, the required area is

$$23{\cdot}5 \times 17{\cdot}6 \text{ square inches,}$$

$$= \mathbf{413{\cdot}6 \text{ sq. in.}}$$

Ex. 13. *A rectangular plot of ground whose length is 55 yards and breadth 47 yards 2 feet has a rectangular lawn 51 yards 1 foot by 44 yards laid out in its central portion, and the remainder is turned into a garden. Find the area of the garden in square feet.*

Area of plot $= 55 \times 3 \times 47\frac{2}{3} \times 3$ sq. ft.
$= 165 \times 143$ sq. ft.
$= 23{,}595$ sq. ft.

Area of lawn $= 51\frac{1}{3} \times 3 \times 44 \times 3$ sq. ft.
$= 154 \times 132$ sq. ft.
$= 20{,}328$ sq. ft.

$\therefore$ area of garden $= 23{,}595 - 20{,}328 = \mathbf{3267}$ sq. ft.

In this case we might have avoided the long multiplication as follows:

Area of garden $= (165 \times 143) - (154 \times 132)$ sq. ft.
$= (11 \times 15 \times 11 \times 13) - (11 \times 14 \times 11 \times 12)$ sq. ft.
$= 363(5 \times 13 - 14 \times 4)$ sq. ft., taking out the common factor $11 \times 11 \times 3 = 363$,
$= 363 \times 9 = \mathbf{3267}$ sq. ft.

as before.

15. Area or Square Measure. We have seen from Prob. 1 that one square inch contains $10 \times 10 = 100$ small squares, each of whose sides is $\frac{1}{10}$ inch. In the same way, a square foot contains $12^2 = 12 \times 12 = 144$ square inches; a square yard $3^2 = 9$ square feet; a square metre $10^2 = 100$ square decimetres, and so on. This explains the following tables, which should be known thoroughly.

British Area Measure.

12^2 or	144	sq. inches	$= 1$ sq. foot
3^2 or	9	sq. feet	$= 1$ sq. yard
$(5\frac{1}{2})^2$ or	$30\frac{1}{4}$	sq. yards	$= 1$ sq. pole
	40	sq. poles	$= 1$ rood
	4	roods	$= 1$ acre
	640	acres	$= 1$ sq. mile

Since 22 yards $= 1$ chain, we have also

22^2 or 484 sq. yards $= 1$ sq. chain,
10 sq. chains or 4840 sq. yd. $= 1$ acre.

Note that the only new terms used are *rood* and *acre*, these being applicable only to area.

In the Metric system, the **square metre** is the unit used for measuring ordinary areas, whilst for land measurement the unit generally employed is the **square dekametre**, which is called an **are**. For large areas, the **hectare** is often used. Thus

1 are = 1 square dekametre = 100 square metres,

100 ares = 1 hectare,

and 100 hectares = 1 square kilometre.

Ex. 14. *The area of a rectangular plot of ground is* 3·75 *acres, and one side measures* 165 *yards. Find the length of the other side in chains.*

165 yards = 165 ÷ 22 = 7·5 chains,

and 3·75 acres = 3·75 × 10 = 37·5 sq. chains.

If, therefore, the length of the unknown side be denoted by l chains, then

$$7{\cdot}5 \times l = 37{\cdot}5.$$

Divide both sides by 7·5 ; then

$$l = \frac{37{\cdot}5}{7{\cdot}5} = \frac{375}{75} = \frac{15}{3} = \mathbf{5} \text{ chains.}$$

Ex. 15. *Find the area in ares of a rectangle measuring* 145 *decimetres by* 98 *decimetres.*

145 decimetres = 14·5 metres,

and 98 decimetres = 9·8 metres ;

∴ the area in square metres = 14·5 × 9·8 = 142·1.

Note that this product might have been simply found as follows :

Since 9·8 is near 10, we may write it 10 − 0·2, so that

$$\begin{aligned} 14{\cdot}5 \times 9{\cdot}8 &= 14{\cdot}5 \times (10 - 0{\cdot}2) \\ &= 14{\cdot}5 \times 10 - 14{\cdot}5 \times 0{\cdot}2 \\ &= 145 - 2{\cdot}9 \\ &= 142{\cdot}1, \text{ as before.} \end{aligned}$$

Now 1 are = 100 sq. metres,

$$\begin{aligned} \therefore\ 142{\cdot}1 \text{ sq. metres} &= 142{\cdot}1 \div 100 \text{ ares} \\ &= 1{\cdot}421 \text{ ares.} \end{aligned}$$

Hence, the area correct to two places is **1·42** ares.

16. Important relations between British and Metric Units of Area. There are a few important relations between the two systems of area measurement which we shall now find, and the results should be remembered for future use.

Ex. 16. *Find correct to two places the number of square centimetres equivalent to a square inch, and use the result to find the equivalent of an acre in ares.*

Imagine a square each of whose sides is exactly one inch long; then since one inch is equivalent to 2·54 centimetres, the area of the square in sq. cm. will give the required result.

Hence the number of sq. cm. equivalent to a sq. in. $= 2{\cdot}54 \times 2{\cdot}54$, which, by contracted multiplication, gives 6·452 to three places.

Hence **one square inch contains 6·45 square centimetres.**

The determination of the equivalent of an acre in ares is a little longer.

Now $\quad 1 \text{ acre} = 4840 \text{ sq. yards},$

and $\quad 1 \text{ sq. yard} = 9 \times 144 = 1296 \text{ sq. inches}.$

Turning each sq. in. into its equivalent in sq. cm., and as the result is required correctly to two places, we must use 6·452 as the equivalent, we get

$$1 \text{ sq. yard} = 1296 \times 6{\cdot}452 \text{ sq. cm.}$$

Hence 1 acre or 4840 sq. yards $= 4840 \times 1296 \times 6{\cdot}452$ sq. cm.

$$= 6{,}272{,}640 \times 6{\cdot}452 \quad \text{sq. cm.}$$

Now, again,

$$1 \text{ metre} = 100 \text{ cm.};$$

$$\therefore\ 1 \text{ sq. metre} = 100^2 \text{ or } 10{,}000 \text{ sq. cm.},$$

and $\quad 1 \text{ are} = 100 \text{ sq. metres}$

$$= 1{,}000{,}000 \text{ sq. cm.}$$

$$\therefore\ 1 \text{ acre} = 6{,}272{,}640 \times 6{\cdot}452 \div 1{,}000{,}000 \text{ ares}$$

$$= 6{\cdot}27264 \times 6{\cdot}452 \text{ ares.}$$

Multiplying this out by the contracted method to two places, we get 40·47 ares.

Hence one acre contains **40·47** ares.

17. Area of a Parallelogram. The sides of a parallelogram are not at right angles to each other as in the rectangle, so that we

cannot measure two adjacent sides as the length and breadth. The area may however be found quite simply.

PROBLEM 2. *To find a rule for determining the area of a parallelogram.*

On a piece of squared paper, draw a parallelogram ABCD (Fig. 17) having the angle D less than a right angle. Draw AE

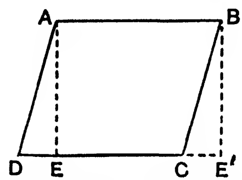


FIG. 17.—Area of a parallelgram.

perpendicular to DC, and BE′ perpendicular to DC produced. Then DC = AB = EE′; from DC and EE′ take away the common part EC; then DE = CE′. Also EA = E′B and DA = CB, so that from geometry we know that the triangle DEA is equal to the triangle CE′B.

Hence the area of $\triangle$DEA = area of $\triangle$CE′B.

To each add the area of the figure EABC, then

area of ($\triangle$DEA + fig. EABC) = area of ($\triangle$CE′B + fig. EABC),

i.e. area of parallelogram ABCD = area of rectangle ABE′E.

But area of rectangle ABE′E = AB $\times$ EA,

so that area of parallelogram = AB $\times$ EA = DC $\times$ EA.

Hence the **area of a parallelogram is measured by the product of the lengths of one side and the perpendicular distance between that side and the opposite side.**

If we make AB = 39 units, AE = 24 units and AD = 26 units, taking the side of a small square as a unit, then CE′ = DE = 10 units by counting the sides of small squares.

The area of the parallelogram is $39 \times 24 = 936$ unit squares.

Now draw the parallelogram so that DA is horizontal, and count the number of units in the perpendicular distance between DA and BC. It is 36, so that, taking DA as base, the area $= 26 \times 36 = 936$ unit squares as before.

18. Area of a Triangle. From the area of a parallelogram we can easily deduce the area of a triangle.

PROBLEM 3. *To find the area of a triangle.*

On a piece of squared paper, draw a triangle ABC having a right angle at A, and the side AB horizontal.

Draw CD parallel to AB, and BD parallel to AC (Fig. 18); then ACDB is a rectangle whose area $=$ AB $\times$ AC.

But $\triangle$ABC $= \frac{1}{2}$ rectangle;

$$\therefore \text{ area of } \triangle ABC = \tfrac{1}{2}(AB \times AC).$$

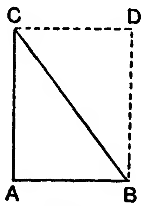


FIG. 18.

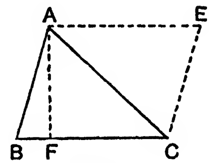


FIG. 19.

If we regard AB as the base, then AC is the altitude of the triangle.

Next draw *any* triangle ABC having BC horizontal. Draw AF perpendicular to BC; AE parallel to BC and CE parallel to BA

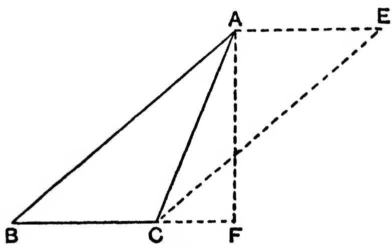


FIG. 20.

(Figs. 19 or 20). BAEC is thus a parallelogram whose area $=$ BC $\times$ AF.

But $\triangle$BCA $= \frac{1}{2}$ parallelogram BAEC;

$$\therefore \text{ area of } \triangle BCA = \tfrac{1}{2}(BC \times AF).$$

Again, BC is the base and AF the altitude.

Note that in Fig. 19 the foot of the perpendicular from A lies within the base BC, whilst in Fig. 20 it lies in BC produced. Hence in all cases, the **area of a triangle is measured by the product $\frac{1}{2}$(base $\times$ altitude).**

Ex. 17. *The area of a triangle standing on a base 96·8 yards long is one-seventh that of a parallelogram whose base is 9 chains 52 links and whose altitude is 6 chains 65 links. Find the altitude of the triangle in yards.*

$$9 \text{ chains } 52 \text{ links} = 9{\cdot}52 \text{ chains} = 9{\cdot}52 \times 22 \text{ yards},$$

and $\quad 6 \quad ,, \quad 65 \quad ,, \quad = 6{\cdot}65 \quad ,, \quad = 6{\cdot}65 \times 22 \quad ,,$

$\therefore$ By Prob. 2,

$$\text{area of parallelogram} = 9{\cdot}52 \times 22 \times 6{\cdot}65 \times 22 \text{ sq. yd.}$$

Hence, area of triangle $= \frac{1}{7}(9{\cdot}52 \times 22 \times 6{\cdot}65 \times 22)$ sq. yd.

Suppose the unknown altitude of the triangle be h yd., then its area, by the above rule, is $\frac{1}{2} \,.\, 96{\cdot}8 \times h = 48{\cdot}4h$ sq. yd.;

$$\therefore \; 48{\cdot}4h = \tfrac{1}{7}(9{\cdot}52 \times 22 \times 6{\cdot}65 \times 22),$$

and

$$h = \frac{9{\cdot}52 \times 22 \times 6{\cdot}65 \times 22}{7 \times 48{\cdot}4} \text{ yd.}$$

$$= 9{\cdot}52 \times 0{\cdot}95 \times 10 = \mathbf{90{\cdot}44} \text{ yd.}$$

19. The Theorem of Pythagoras. An important relationship between the lengths of the sides of a right-angled triangle was discovered by Pythagoras, a Greek who lived in Sicily, B.C. 570-500. We shall now see what this relationship is.

Ex. 18. *Find, by measurement, the relation between the lengths of the sides of a right-angled triangle.*

Draw a good-sized triangle ABC having a right angle at C; then measure the sides carefully, and calculate the areas of the squares on these sides. Thus, writing CA^2 for the area of the square on CA, and similarly for the other sides.

Suppose $BC = 2{\cdot}8$ in., then $BC^2 = 7{\cdot}84$ sq. in.

$CA = 4{\cdot}5$ in., then $CA^2 = 20{\cdot}25$ sq. in.;

$\therefore \; BC^2 + CA^2 = 7{\cdot}84 + 20{\cdot}25 = 28{\cdot}09$ sq. in.

Also, by measurement,

$AB = 5{\cdot}3$ in., so that $AB^2 = 28{\cdot}09$ sq. in.

Hence in this case, $\quad AB^2 = BC^2 + CA^2$.

Note that AB is the hypotenuse, and BC, CA, the arms of the right angle.

Draw several other right-angled triangles, and see whether the same is true for each of them.

EXERCISES 3A.

ABC is a triangle having a right angle at C; draw the triangle in each of the following cases, measure the third side and calculate the values of BC^2+CA^2 and AB^2.

1. BC = 3·5 in., CA = 1·2 in.
2. BC = 4 in., CA = 0·9 in.
3. AB = 6·1 in.; CA = 1·1 in.
4. BC = 5·6 in., CA = 3·3 in.
5. AB = 3·4 in., BC = 1·6 in.
6. AB = 8·9 cm., BC = 3·9 cm.
7. AB = 9·7 cm., BC = 7·2 cm.
8. AB = 10·1 cm., BC = 9·9 cm.
9. AB = 16·9 cm., CA = 11·9 cm.
10. BC = 16·8 cm., CA = 9·5 cm.

11. On a piece of stout paper draw a triangle ABC having a right angle at C; describe squares on each of the sides (Fig. 21). Suppose BCHK to be the larger of the squares on BC and CA. Find its centre O by drawing the diagonals and draw POQ parallel to AB, and ROS perpendicular to POQ. Cut along these lines and along BC, thus making four pieces, *a*, *b*, *c*, *d*. Stick these on the square ABDE as indicated, and then cut off the square ACFG along AC. This square will then fit into the uncovered space in ABDE. Thus the sum of the areas of the squares on BC, CA make up the area of the square on AB. (This method of dissection was first given by Henry Perigal in 1830.)

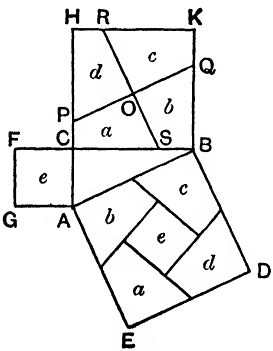


FIG. 21.—Perigal's dissection.

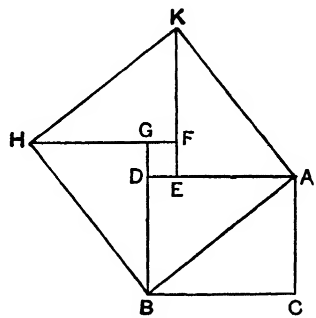


FIG. 22.—Squared paper dissection.

12. On a piece of squared paper draw a triangle ABC right-angled at C (Fig. 22). Draw BG = BC and parallel to CA; AD equal and parallel to BC, and on it make AE = CA. From E draw EK = BC and parallel to CA, and on it mark KF = CA. Draw FH equal and parallel to BC. This will pass through G, since DG = FE = BC − CA. Join AK, KH, HB. Shew that this is the square on AB, and find its area by determining the number of small squares in the four triangles ABD, BHG, HKF, KAE, and the small square DEFG. Compare this with the sum of the areas of the squares on BC, CA.

20. The General Case. We shall now see whether the relation of Ex. 18 is true of all right-angled triangles.

PROBLEM 4. *To find the general relation between the sides of a right-angled triangle.*

In Fig. 22, let the lengths of BC, CA and AB be denoted by a, b, c respectively; then the area of the triangle ABC $=\frac{1}{2}ab$. But each of the four triangles ABD, BHG, HKF, KAE is equal to ABC, and DG = GF = FE = ED = BC − CA $= a - b$;

∴ area of square ABHK $= 4 \cdot \frac{1}{2}ab + (a-b)^2 = a^2 + b^2$.

But area of square ABHK $= c^2$;

$$\therefore\ c^2 = a^2 + b^2,$$

i.e. **in a right-angled triangle the square on the hypotenuse = sum of squares on the other sides.**

This is the famous theorem of Pythagoras, and its applications are exceedingly important.

Ex. 19. *The sides of a rectangle are 2 ft. and 7 in. long respectively; find the length of the diagonal.*

If we cut the rectangle along one of its diagonals, we have a right-angled triangle, having the diagonal for its hypotenuse. Let the length of this diagonal be h in., then

$$h^2 = 24^2 + 7^2 = 576 + 49 = 625;$$

$$\therefore\ h = \sqrt{625} = 25 \text{ in.}$$

Ex. 20. BC, CA *are two sides of a rectangular field,* BC *being* 11 *chains* 88 *links*, and CA 7 *chains* 65 *links. Two men start from* B, *to walk to* A; *one walks along the diagonal* AB *and the other along the sides* BC, CA. *Find how many yards farther this man goes.*

Here we must first find the length of the diagonal.

Now 11 chains 88 links = 1188 links,

and 7 „ 65 „ = 765 „

∴ if AB be h links long,

$$\begin{aligned} h^2 &= 1188^2 + 765^2 \\ &= 1{,}411{,}344 + 585{,}225 \\ &= 1{,}996{,}569; \end{aligned}$$

$$\therefore\ h = \sqrt{1{,}996{,}569} = 1{,}413 \text{ links,}$$

on extracting the square root by the ordinary arithmetical process.

But $\quad BC+CA=1188+765=1953$ links.

$\therefore$ difference in distances walked

$$=1953-1413=540 \text{ links}=5{\cdot}4 \text{ chains}$$
$$=5{\cdot}4\times 22=\mathbf{1188} \text{ yards.}$$

Ex. 21. ABC *is a triangle in which* AB $=8$ *ft.* 10 *in.*, BC $=10$ *ft.* 3 *in.*, *and* CA $=5$ *ft.* 5 *in.* *The perpendicular from* A *to* BC *meets it in* D ; *find the lengths of* BD, DC, AD, *and the area of the triangle in square yards.*

Let the length of BD (Fig. 23) be x in., then DC $=(123-x)$ in.

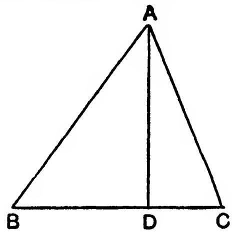


FIG. 23.

Now BD divides the triangle into two right-angled triangles, ABD and ACD ; hence

$$AB^2=BD^2+AD^2, \text{ and } CA^2=CD^2+AD^2,$$

i.e. $\quad 106^2=x^2+AD^2$, and $65^2=(123-x)^2+AD^2$,

or $\quad 106^2-x^2=AD^2$, and $65^2-(123-x)^2=AD^2$.

We thus have two expressions for AD^2, which must, therefore, be equal. Hence

$$106^2-x^2=65^2-(123-x)^2$$
$$=65^2-123^2+246x-x^2 ;$$

hence
$$246x=106^2-65^2+123^2$$
$$=11236-4225+15129=22140 ;$$
$$\therefore\ x=22140\div 246=90 \text{ in.}$$
$$\therefore\ CD=123-x=123-90=33 \text{ in.}$$

As x is now known, the value of AD can be found from either of the expressions above. Taking the simpler,

$$AD^2=106^2-x^2=106^2-90^2.$$

This being the *difference* of two squares, we may use the identity

$$p^2-q^2=(p+q)(p-q),$$

which should always be done in such cases.

Hence $\quad \text{AD}^2 = (106+90)(106-90) = 196 \times 16 = 14^2 \times 4^2$;

$$\therefore \text{AD} = 14 \times 4 = 56 \text{ in.}$$

$\therefore$ BD $= 90$ in. $= 7$ ft. 6 in., DC $= 33$ in. $= 2$ ft. 9 in., and

$$\text{AD} = 56 \text{ in.} = 4 \text{ ft. } 8 \text{ in.}$$

Now the area of the triangle $= \frac{1}{2}$. BC . AD $= \frac{1}{2} \times 123 \times 56$ sq. in.

$$= 3444 \text{ sq. in.} = \frac{3444}{144 \times 9} \text{ sq. yd.} = \mathbf{2{\cdot}66} \text{ sq. yd.}$$

21. Area of a Trapezium. Knowing now how to find the area of a triangle, it is quite easy to determine the area of a trapezium.

Ex. 22. ABCD *is a plot of ground in the shape of a trapezium,* AB, CD *being the parallel sides. If* CD $=34$ *chains* 42 *links,* AB $=36$ *chains* 83 *links, and the perpendicular distance between these sides is* 24 *chains, calculate the area of the plot in acres.*

Let Fig. 24 represent the plot. Draw the diagonal BD, then the trapezium is divided into two triangles, DAB, BCD; hence its

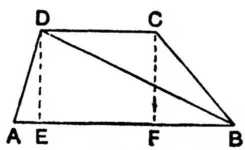


FIG. 24.—Area of a trapezium.

area is the sum of the areas of these triangles. Draw DE, CF perpendicular to AB, then ED $=$ FC $= 24$ chains, and

$$\text{area of } \triangle\text{DAB} = \tfrac{1}{2} \,.\, \text{AB} \times \text{ED} = \tfrac{1}{2} \,.\, 36{\cdot}38 \times 24 \text{ sq. chains,}$$

$$\text{area of } \triangle\text{BCD} = \tfrac{1}{2} \,.\, \text{CD} \times \text{CF} = \tfrac{1}{2} \,.\, 34{\cdot}42 \times 24 \text{ sq. chains.}$$

$\therefore$ area of trapezium ABCD

$$= (\tfrac{1}{2} \,.\, 36{\cdot}83 \times 24) + (\tfrac{1}{2} \,.\, 34{\cdot}42 \times 24) \text{ sq. chains}$$

$$= \tfrac{1}{2} \,.\, 24(36{\cdot}83 + 34{\cdot}42) \text{ sq. chains}$$

$$= 12 \times 71{\cdot}25 = 855 \text{ sq. chains}$$

$$= \mathbf{85{\cdot}5} \text{ acres.}$$

PROBLEM 5. *To find the area of any trapezium.*

Let the lengths of AB, CD (Fig. 24), be denoted by a, b respectively, and the perpendicular distance between these sides be h, then the area of the trapezium $=$ area of ADB $+$ area of CBD

$$= \tfrac{1}{2}ah + \tfrac{1}{2}bh = \tfrac{1}{2}h(a+b).$$

We might consider the trapezium to be made up of the two right-angled triangles AED, BFC and the rectangle EDCF, so that its area
$=\frac{1}{2}h \,.\, \mathrm{AE}+\frac{1}{2}h \,.\, \mathrm{FB}+hb=\frac{1}{2}h(\mathrm{AE}+\mathrm{FB}+2b)=\frac{1}{2}h(a+b)$, as before.

Hence **the area of a trapezium is measured by the product,**

$\frac{1}{2}$(sum of parallel sides) × (their distance apart).

When the four sides are given, it is necessary first to find the distance between the parallel sides. This may be done by an application of the theorem of Pythagoras. The method is best illustrated by a numerical example.

Ex. 23. ABCD *is a trapezium in which* AB, CD *are the parallel sides. Find its area in square yards when* AB $=7{\cdot}5$ *ft.*, BC $=20$ *ft.*, CD $=28{\cdot}5$ *ft., and* DA $=13$ *ft.*

Let ABCD (Fig. 25) be the trapezium; draw AE, BF perpendicular to CD meeting it at E, F respectively.

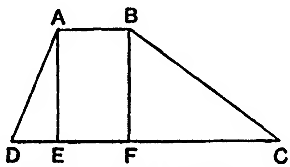


FIG. 25.—Area of a trapezium.

Suppose DE $=x$ ft. long, and AE $=$ BF $=h$ ft. long, then since ADE is a right-angled triangle,

$$\mathrm{AE}^2=\mathrm{AD}^2-\mathrm{DE}^2 \quad i.e. \quad h^2=13^2-x^2.$$

Similarly, since BCF is a right-angled triangle, and

$$\mathrm{FC}=\mathrm{DC}-\mathrm{DF}=28{\cdot}5-(7{\cdot}5+x)=21-x,$$

$$\therefore \ \mathrm{BF}^2=\mathrm{BC}^2-\mathrm{FC}^2 \quad i.e. \quad h^2=20^2-(21-x)^2.$$

Hence these two expressions for h^2 must be equal, so that

$$20^2-(21-x)^2=13^2-x^2;$$

$$\therefore \ 400-441+42x-x^2=169-x^2,$$

from which $$42x=169-400+441=210;$$

$$\therefore \ x=5,$$

and $$h^2=13^2-x^2=13^2-5^2=18 \,.\, 8=12^2;$$

$$\therefore \ h=12.$$

Hence, applying the above rule, the area of the trapezium

$$=\tfrac{1}{2} \,.\, 12 \,.\, (28{\cdot}5+7{\cdot}5)=6\times 36 \text{ sq. ft.}$$

$$=\mathbf{24} \text{ sq. yards.}$$

EXERCISES 3B.

Rectangles.

Find the area in the units stated of each of the following rectangles, whose linear dimensions are given:

1. 110 yd. by 99 yd., in acres.

2. 8 chains 25 links by 3 chains 54 links, in acres.

3. 18 ft. by 4 ft. 6 in., in square yards.

4. 3·87 m. by 5·4 m., in square yards, taking 6·45 sq. cm. as the equivalent of one square inch.

5. 539·6 m. by 453 m., in acres, taking one acre to be equivalent to 40·47 ares.

6. The area of a rectangle is $1\frac{1}{4}$ acres; one side measures 2 chains 50 links. Find the length of the other side.

7. The width of a rectangle is 7 ft. 6 in. and its area is 16·5 sq. yd. Find its length.

8. One side of a rectangle measures 3 ft. 4 in. and its area is 43·86 sq. dm. Find the length of the other side in inches, taking 6·45 sq. cm. as the equivalent of 1 sq. in.

9. A rectangular field covers an area of $2\frac{1}{2}$ acres, and one side of it is 269·8 metres long. Taking 40·47 ares as the equivalent of 1 acre, find the length of the other side in metres.

10. What will be the ground rent of a rectangular plot of ground, measuring 74 ft. 3 in. by 29 ft. 4 in., at the rate of £50 per acre? (C.S.)

11. A new core-shop floor is to be covered with cast-iron plates. Its length is 30 ft. 6 in. and its width 15 ft. 9 in. What will be the cost at 18s. 6d. per sq. yd.? (U.L.C.I.)

12. The outside measurements of a casement window of one pane are 3 ft. 9 in. and 1 ft. 9 in. The frame is $1\frac{1}{2}$ in. in width all round. Find the area of glass required.
If the window were divided into six panes by one vertical and two horizontal strips of wood an inch wide, by how much would the area of the glass be decreased? (C.S.)

13. What length of flooring-board $5\frac{1}{2}$ in. wide is required for a floor 15 ft. 4 in. long by 13 ft. 9 in. wide, the boards being laid parallel to the longer side?

14. Find the whole area in square feet of the walls of a room 20 ft. 6 in. long, 16 ft. 4 in. broad and 15 ft. high, deducting 120 square feet for doors and windows. What will be the cost of painting the walls at $\frac{1}{4}$d. per square foot? (C.S.)

15. The total area of the walls of a room is 450 sq. ft., the area of the floor is 156 sq. ft. Find the height of the room if the length is 12 ft. (U.L.C.I.)

16. A room, 18 feet long and 16 feet wide, has a stained border one foot six inches wide; the rest of the floor is to be covered with a Turkey carpet at 18 shillings per square yard. What will be the cost of the carpet? (C.S.)

17. Find the number of pieces of paper, each 12 yd. long and 22 in. wide, required for papering the walls of a room 25 ft. 3 in. long, 18 ft. 9 in. broad, and 13 ft. 6 in. high. (C.S.)

18. Part of a rectangular floor measuring 24 ft. by 15 ft. is covered by a carpet measuring 20 ft. by 11 ft., and the remainder of the floor is covered by linoleum. If the carpet costs 31s. 6d. per sq. yd. and the linoleum 12s. 3d. per sq. yd., find to the nearest penny the total cost of covering the floor. (C.S.)

19. A room is 31 feet long and 22 feet wide. Find the cost of carpeting it, leaving a border 2 feet 6 inches wide all round to be stained, if the carpet is 26 inches wide and costs 8s. 9d. per yard. Find also the cost of staining the border at 1s. 9d. per square yard. (L.M.)

20. It is required to cover with linoleum the floor of a room in France measuring 12·7 metres by 7·62 metres. The linoleum is purchased in England, and the total cost is £26 0s. 10d. Taking 1 yard = 0·9144 of a metre, find the price paid per sq. yd.

21. How many metres of carpet 60 cm. wide are required to cover a floor 11 metres long by 7 metres wide, leaving a border 20 cm. wide all round? (L.S.)

22. A rectangular field is measured by a tape whose length is supposed to be 66 feet, but which has actually shrunk 6 inches. According to this tape, the field is 220 yards long and 55 wide. Find to the nearest square yard by how much the area of the field is overestimated. (L.S.)

23. A pathway 3 feet 6 inches wide surrounding a rectangular lawn 71 feet 9 inches by 57 feet 9 inches is paved with tiles 7 inches square. Find the cost of the tiles at £1 7s. per gross.

24. Wallpaper is generally made in pieces 12 yards long and 21 inches wide. To find the number of pieces required for a room, a decorator's rule is to divide the area in sq. ft. of the walls to be covered by 54. What fraction of each piece does this allow for waste? Apply the rule to find the number of pieces required for a room 42 ft. 7 in. long by 27 ft. 5 in. wide by 15 ft. high, allowing 28 per cent. of the area to be occupied by fireplace, doors and windows.

Square Root.

25. The area of a square field is $2\frac{1}{2}$ acres; find the length of its side in yards.

26. Find the side of a square courtyard whose area is 9 acres 4401 sq. yd.

27. How many metres are there in each side of a square field containing 778·41 ares ?

28. A square field contains 15 acres. Find, to the nearest yard, the length of fencing required to enclose it. (C.S.)

29. A square sheet of metal has an area of 549,081 sq. in.; find the length of one edge. (U.L.C.I.)

30. Find the length of the side of a square whose area is equal to that of a rectangle measuring 14 ft. 1 in. by 4 ft. 1 in.

31. A square field contains 17,653 square yards. Find the length of a side correct to the nearest foot. What is the least number of square yards by which the area of the field, still remaining square, should be increased for a side to be an exact number of yards ?

Hence find the two factors of 17,653. (L.S.)

32. Find the area in hectares of a square field whose diagonal is 450 metres in length, it being given that a hectare is a square whose side is 100 metres. (C.S.)

33. The side of a square whose area is 0·27 are is given as $5\frac{10}{51}$ metres. Shew that the actual length differs from this by less than 0·1 of a millimetre.

34. The side of a square equal in area to a rectangle 7 yd. 2 ft. long by 5 yd. 2 ft. wide is taken as $19\frac{24}{31}$ ft. Prove that this is greater than the actual value by a quantity less than 0·006 of an inch.

Triangles and Trapeziums.

35. A triangle is 1 yd. 1 ft. 4 in. high and its base is 2 yd. 2 ft. 4 in. long. Find its area in square centimetres, taking 2·54 cm. to an inch.

36. The base and altitude of a triangle are 309·6 cm. and 275·4 cm. respectively. Its area measured in British units is 5·1 sq. yd. Calculate the number of square centimetres equivalent to a square inch.

37. ABCD is a trapezium in which AB, CD are the parallel sides. If AB = 584·3 in., CD = 375·7 in., and the perpendicular distance between them is 2 ft. 3 in., find the area of the figure in square yards.

38. A courtyard shaped like a trapezium has its parallel sides 49 ft. and 68 ft. long respectively, their distance apart being 21 ft. How many tiles each 13 in. by 9 in. will be required to cover it ?

39. The area of a trapezium is 2 sq. ft. 26·5 sq. in.; the lengths of the parallel sides are 1 ft. 7 in. and 1 ft. 6 in. respectively; find their perpendicular distance apart.

40. A plot of ground is represented in a plan by a figure FEBCGDA, in which ABCD is a trapezium having AB, CD for its parallel sides, ABEF is a rectangle and CDG is a triangle. AB = 1 ft. 9 in., BE = CD = 2 ft. 11 in., perpendicular distance between AB, CD = 8·75 in., and perpendicular distance of G from CD = 1 ft. 2 in. The shape is to be altered to a square having an equal area. Find the side of this square.

Theorem of Pythagoras.

41. The diagonal of a rectangle 10 ft. long measures 14 ft. 1 in. Find the width of the rectangle.

42. One side of a right-angled triangle is 4 yards long, and its hypotenuse is just one inch longer. Find the length of the third side and the area of the triangle.

43. The lengths of the sides of a lawn tennis-court are 78 feet and 36 feet; calculate the length of the diagonal to the nearest foot. (C.S.)

44. How far up a vertical wall will a ladder 50 ft. long reach if its foot be placed 14 ft. away from the wall?

45. A right-angled triangle has its shorter sides 2·5 in. and 3 in. respectively. Calculate to two places of decimals the length of the remaining side and also the area of the triangle. (U.L.C.I.)

46. A shed with a roof sloping from back to front is 8 ft. wide and 10 ft. long. The height of the front wall is 5 ft. and that of the back wall 11 ft. Find how many yards of felt 2 ft. wide will be required to cover the roof completely.

47. An estate is shewn on a plan as a four-sided figure ABCD in which BC = CD = 65 cm., AD = 30 cm., ∠ADC = 90°, and the perpendicular from B to CD = 60 cm. Find the area of the estate in hectares, if the scale of the plan is 1 cm. to 5 metres.

48. The roof of a house has unequal slopes, whose lengths from the ridge to the eaves are 22·5 ft. and 30 ft. respectively. Find the depth of the house from back to front if the ridge is 18 ft. above the eaves.

49. Of five towns, B is 14·4 miles due east of A, C is 6 miles north of B, D is 2·1 miles east of C, and E is 2·8 miles north of D. In travelling from A to E, how many miles would be saved if the journey could be made by taking two straight roads joining A and C, and C and E respectively, instead of taking straight roads from A to B, B to C, C to D, and D to E? Calculate the shortest distance between A and E.

In each of the following examples, find the length of the perpendicular AD and the area of the triangle ABC, D being the point in BC where the perpendicular to it from A meets it.

50.	AB = 20 in.,	BC = 21 in.,	CA = 13 in.
51.	AB = 17 in.,	BC = 44 in.,	CA = 39 in.
52.	AB = 41 in.,	BC = 51 in.,	CA = 58 in.
53.	AB = 97 ft.,	BC = 219 ft.,	CA = 170 ft.
54.	AB = 85 ft.,	BC = 48 ft.,	CA = 91 ft.
55.	AB = 250 yd.,	BC = 339 yd.,	CA = 137 yd.
56.	AB = 87 yd.,	BC = 76 yd.,	CA = 65 yd.
57.	AB = 29 cm.,	BC = 240 cm.,	CA = 221 cm.

58. A triangle has sides of lengths 17 ft., 25 ft., 28 ft. Find the least altitude and the area, and find the area of the triangle cut off from it by the straight line which bisects the longest side at right angles. (C.P.)

Miscellaneous.

59. A square plot of ground is surrounded by a gravel path 5 feet wide, the outside boundary of the gravel also forming a square. It is desired to double the width of the path, and it is found that $1\frac{1}{4}$ times as much gravel is required for the extension as for the original path. Find the length of the side of the plot of ground. (L.S.)

60. A rectangular garden is 11 yards longer than it is wide and occupies 0·15 acre. Surrounding the garden is a path of uniform width, and the length of fencing edging the outer boundary of the path is 126 yards. Find the dimensions of the garden and the width of the path.

61. Two lawns, one a rectangle, the other a square, are of the same area, 484 sq. yd. Four times the perimeter of the rectangle is equal to five times that of the square. Find the length and breadth of the rectangle. (L.M.)

62. A flat rectangular roof is to be covered with asphalte. One side is 2 ft. shorter than the other. If the shorter side of the roof were increased by 57 ft. and the longer side reduced by 24 ft. the area would remain the same. Calculate the length of each side. (U.L.C.I.)

63. ABCD is a trapezium having the non-parallel sides AB, CD 17 cm. and 25 cm. long respectively, and BC = 2AD = 56 cm.; find the area of the figure.

64. The parallel sides of a trapezium are $(x-11)$ ft. and $(x-7)$ ft. respectively; their distance apart is x ft. If its area be equal to that of a square whose side is $(x-18)$ ft., find x.

65. ABCD is a trapezium having AB, CD as its parallel sides. If AB = 9 ft. 11 in., BC = 28 ft. 4 in., CD = 49 ft. 4 in., and DA = 18 ft. 3 in., find the perpendicular distance between AB and CD, and the area of the figure in square yards.

66. The parallel sides of a trapezium are 1·8 ft. and 7 ft. long respectively, and their distance apart is 4·5 ft. Find the lengths of the non-parallel sides when they differ in length by 2·4 in.

67. The lengths of the sides of a right-angled triangle ABC are: AB = $(x-1)$ inches, AC = $(x-2)$ inches, and BC = 3·2 inches, $\angle$C being the right angle. Find the value of x. (U.L.C.I.)

68. The diagonal of a large rectangular field is 50 chains, and its area is 120 acres. Find its length. (L.M.)

69. ABCD is the plan of a rectangular field in which AB = 21 chains, and BC = 22·9 chains. E and F are points in DA, BC respectively such that DE = BF = 2·9 chains. Find the area of the diagonal strip BFDE, and the distance between EB and DF.

70. In a street 49 feet wide, a ladder reaches 63 feet up the vertical wall of a house on one side, and when turned over, keeping the foot in the same place, reaches 56 feet up the vertical wall of the house opposite. Find how far the foot of the ladder is from this wall, and the length of the ladder.

71. ABC is a triangle having a right angle at C; from a point D in AB, DE, DF are drawn parallel to CA, BC respectively, meeting them in E and F such that DE = DF. If AB = 44·2 cm., and BC = 17 cm., find the lengths of DF and BD.

72. ABCD is a trapezium, and the non-parallel sides AB, CD are each 17 cm. long. The perimeter of the trapezium is 100 cm. and its area is 495 sq. cm. Find the lengths of the parallel sides, and the perpendicular distance between them. (L.S.)

73. ABCD is a trapezium whose perimeter is 118 cm. The parallel sides AB, CD differ in length by 17 cm., AB being the shorter, DA = AB, and BC is 1 cm. longer than AB. Find the area of the trapezium.

CHAPTER IV

AREAS OF RECTILINEAL FIGURES. THE FIELD BOOK. SIMPSON'S RULE

22. Area of a Triangle in Terms of its Sides. In the case of a triangle ABC we shall follow the usual custom in denoting the lengths of the sides BC, CA, AB by a, b, c respectively, and the angles BAC, ABC, BCA by A, B C; the length of a side is denoted by the small letter corresponding to the opposite angle. We shall also denote the area of the triangle by the Greek letter delta, Δ.

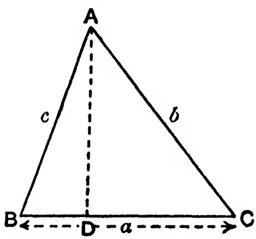


FIG. 26.—Area of a triangle.

In practical work it is often necessary to know how to find the area of a triangle directly from the measurements of its sides, and by a simple algebraic application of the method of Ex. 21 (p. 33), a general rule may readily be found.

PROBLEM 6. *To find the area of a triangle in terms of its sides.*

Let ABC (Fig. 26) be any triangle. Draw AD perpendicular to BC, meeting it in D. Let BD $= x$, then DC $=$ BC $-$ BD $= a - x$. From the right-angled triangle ABD,

$$c^2 = \mathrm{AD}^2 + x^2, \quad \text{or} \quad \mathrm{AD}^2 = c^2 - x^2.$$

Similarly, from the right-angled triangle ACD,

$$\mathrm{AD}^2 = b^2 - (a - x)^2;$$

$$\therefore\ b^2 - (a-x)^2 = c^2 - x^2,$$

or
$$b^2 - a^2 + 2ax - x^2 = c^2 - x^2;$$

$$\therefore\ x = (c^2 + a^2 - b^2)/(2a).$$

Now the altitude AD is given by

$$\text{AD}^2 = c^2 - x^2 = (c+x)(c-x)$$
$$= \left(c + \frac{c^2+a^2-b^2}{2a}\right)\left(c - \frac{c^2+a^2-b^2}{2a}\right)$$
$$= (2ac + c^2 + a^2 - b^2)(2ac - c^2 - a^2 + b^2)/4a^2;$$
$$\therefore\ 4a^2 . \text{AD}^2 = \{(c+a)^2 - b^2\}\{b^2 - (c-a)^2\}$$
$$= (c+a+b)(c+a-b)(b+c-a)(b-c+a).$$

The expression on the right-hand side may be made a little simpler by taking $2s$ as the perimeter of the triangle, *i.e.*

$$2s = a + b + c;$$

then $\qquad 2s - 2a = a + b + c - 2a = -a + b + c.$

Similarly, $\quad 2s - 2b = a - b + c \quad$ and $\quad 2s - 2c = a + b - c;$

$$\therefore\ 4a^2 . \text{AD}^2 = 2s(2s-2b)(2s-2a)(2s-2c)$$
$$= 16s(s-a)(s-b)(s-c);$$
$$\therefore\ \tfrac{1}{4}a^2 . \text{AD}^2 = s(s-a)(s-b)(s-c).$$

Taking the square root of each side,

$$\tfrac{1}{2}a . \text{AD} = \sqrt{s(s-a)(s-b)(s-c)}.$$

But $\frac{1}{2}a$. AD = area of triangle $= \Delta$;

$$\therefore\ \Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

Hence, **the area of a triangle is measured by** $\sqrt{s(s-a)(s-b)(s-c)}$, **where** a, b, c **are the sides and** s **is the semi-perimeter.**

It is interesting to note that this rule was discovered by Hero, a famous mathematician of Alexandria who lived during the first or second century.

Ex. 24. *Find the acreage of a triangular plot of land whose sides measure* 65, 66·6, *and* 37·6 *chains respectively.*

Using the notation of Problem 6, let $a = 65$, $b = 66{\cdot}6$, and $c = 37{\cdot}6$; then $2s = 65 + 66{\cdot}6 + 37{\cdot}6 = 169{\cdot}2$;

$\therefore\ s = 84{\cdot}6$, $s - a = 84{\cdot}6 - 65 = 19{\cdot}6$, $s - b = 84{\cdot}6 - 66{\cdot}6 = 18$, and $s - c = 84{\cdot}6 - 37{\cdot}6 = 47$.

$$\therefore\ \text{area of plot} = \sqrt{84{\cdot}6 \times 19{\cdot}6 \times 18 \times 47} \text{ sq. ch.}$$
$$= \tfrac{1}{10}\sqrt{846 \times 196 \times 18 \times 47} \text{ sq. ch.}$$
$$= \tfrac{1}{10}\sqrt{18 \times 47 \times 14 \times 14 \times 18 \times 47} \text{ sq. ch.}$$
$$= \tfrac{1}{10} . 18 \times 47 \times 14 = 1184{\cdot}4 \text{ sq. ch.};$$

$\therefore$ area in acres $=$ **118·44.**

EXERCISES 4A.

Find the area of each of the following triangles:

1. $a=15$ in., $b=14$ in., $c=13$ in.
2. $a=17$ in., $b=25$ in, $c=28$ in.
3. $a=43$ ft., $b=61$ ft., $c=68$ ft.
4. $a=77$ ft., $b=74$ ft., $c=25$ ft.
5. $a=57$ ft., $b=82$ ft., $c=89$ ft.
6. $a=34{\cdot}8$ chains, $b=30{\cdot}5$ ch., $c=7{\cdot}3$ ch.
7. $a=b=53$ ft., $c=56$ ft.
8. $a=26$ ft., $b=c=85$ ft.
9. $a=c=22{\cdot}9$ yd., $b=12{\cdot}6$ yd.
10. $a=b=c=12$ yard.
11. $a=b=c=29$ chains.
12. $a=b=c=32{\cdot}4$ chains.
13. $a=5$ in., $b=8$ in., $c=11$ in.
14. $a=13$ in., $b=17$ in., $c=24$ in.
15. $a=29$ in., $b=18$ in., $c=31$ in.
16. $a=57$ ft., $b=52$ ft., $c=53$ ft.
17. $a=73$ ft., $b=74$ ft., $c=21$ ft.
18. $a=9{\cdot}7$ ch., $b=7{\cdot}5$ ch., $c=8{\cdot}6$ ch.
19. $a=56$ yd., $b=c=52$ yd.
20. $a=b=82$ yd., $c=56$ yd.

21. Shew that the area of an equilateral triangle is $\sqrt{3}/4$ times the area of the square described on its side.

22. In an isosceles triangle in which $a=b$, prove that its area is $\frac{1}{4}c\sqrt{c^2+4a^2}$.

23. Find an expression for the area of a triangle whose sides are $a-x$, a, and $a+x$ respectively.
If the area of such a triangle be 84 sq. in., and $a=1$ ft. 2 in., find x.

24. Find the area of a quadrilateral ABCD having AB $=11$ in., BC $=13$ in., CD $=14$ in., DA $=20$ in., and its diagonal CA $=18$ in.

25. ABCD is a quadrilateral such that the angle ADC $=90°$. If AB $=2$ ft. 1 in., BC $=2$ ft. 5 in., CD $=2{\cdot}4$ ft., and the diagonal CA $=3$ ft., find the area of the figure in square feet.

23. Area of any Rectilineal Figure. We can find the area of any plane rectilineal figure by dividing it into triangles and taking sufficient measurements to enable us to calculate the area of each triangle.

Ex. 25. *The plan of a field is given drawn to a scale* 1 *cm. to* 20 *yards. By copying the figure accurately, find from it the actual area of the field in acres.*

Suppose Fig. 27 represents the plan when copied. The actual copying may be readily done by constructing the triangles ABD, BCD by the usual geometrical method. To save space, Fig. 27 is drawn on a much smaller scale than that given.

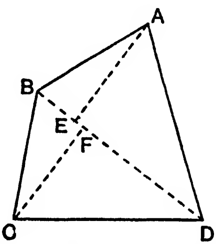


FIG. 27.—Area of a quadrilateral.

Now, to find the total area, we have to find the areas of the two triangles ABD, BCD; their altitudes will therefore be required. Hence, draw AE, CF each perpendicular to BD, and measure very carefully in centimetres the lengths of BD, AE, CF. This may be conveniently done by cutting a narrow strip of squared paper accurately ruled in millimetre squares.

Suppose the actual measurements to be: BD $=8{\cdot}4$ cm., AE $=4{\cdot}8$ cm., CF $=4{\cdot}5$ cm., then the real lengths are:

$$\text{BD}=8{\cdot}4\times 20=168 \text{ yd.,}\quad \text{AE}=4{\cdot}8\times 20=96 \text{ yd.,}$$

and CF $=4{\cdot}5\times 20=90$ yd., since 1 cm. represents 20 yd.

$$\begin{aligned}\therefore \text{ area of ABCD} &= \triangle\text{ABD}+\triangle\text{BCD}=\tfrac{1}{2}\,.\,\text{BD}\,.\,\text{AE}+\tfrac{1}{2}\,.\,\text{BD}\,.\,\text{CF}\\ &=84\times 96+84\times 90 \text{ sq. yd.}\\ &=84(96+90) \text{ sq. yd.}=84\times 186 \text{ sq. yd.}\\ &=\frac{84\times 186}{4840}=\frac{1953}{605}=3{\cdot}23 \text{ acres.}\end{aligned}$$

We might have found the area of each triangle by measuring its three sides and using the rule of Prob. 6 but the method is much longer and more liable to error, since two more measurements are required. It would, however, be a useful exercise to use the method to check the above result.

EXERCISES 4B.

Copy each of the following figures twice its size and by making sufficient measurements, find the actual area represented.

1.

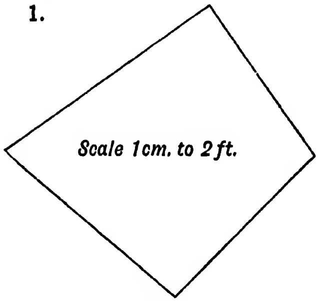


FIG. 28.

2.

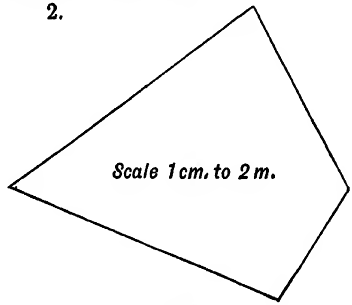


FIG. 29.

3.

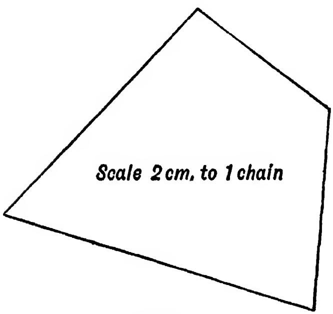


FIG. 30.

4.

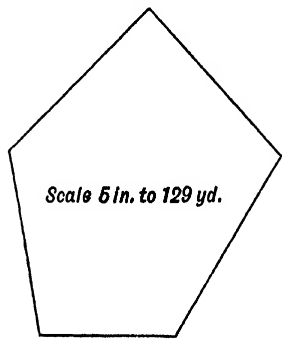


FIG. 31.

5.

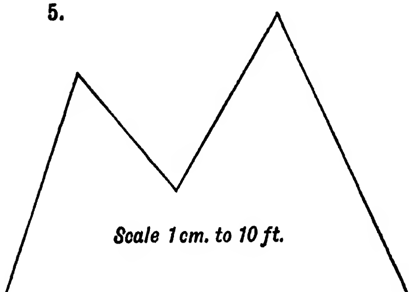


FIG. 32.

6.

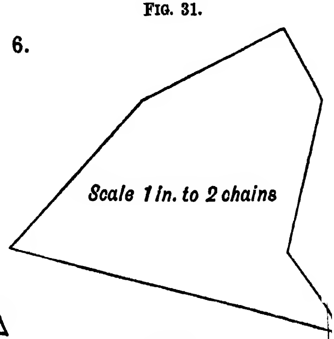


FIG. 33.

24. Areas from given Measurements. When the measurements are given, the rule of Prob. 6 (p. 42) must be employed, or else other lengths must be calculated from which the required area may be found. The methods of the following examples should be carefully studied.

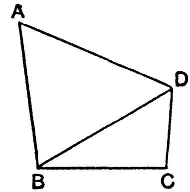


FIG. 34.—Plan of a field.

Ex. 26. ABCD *is a quadrilateral field in which* AB = 39 *chains*, BC = 34 *chains*, CD = 20 *chains*, DA = 45 *chains, and the diagonal* BD = 42 *chains. Calculate the area of the field in acres.*

The plan of the field is shewn in Fig. 34, from which it will be evident that we have to calculate the areas of the two triangles ABD, BCD.

Now the semi-perimeter of $ABD = \frac{1}{2}(39 + 42 + 45) = 63$ chains,

and „ „ „ $BCD = \frac{1}{2}(34 + 20 + 42) = 48$ „

$$\therefore \triangle ABD = \sqrt{63 \times 24 \times 21 \times 18} = 21 \times 36 = 756 \text{ sq. ch.}$$

and $$\triangle BCD = \sqrt{48 \times 14 \times 28 \times 6} = 16 \times 21 = 336 \text{ sq. ch.}$$

$\therefore$ area of whole field $= 756 + 336 = 1092$ sq. ch. $=$ **109·2** acres.

Ex. 27. *A plot of land* ABCDEFGA *is bounded by seven straight edges. The angles* GAB, GBC *are right angles, and* GC *is parallel to* FD. *Find the acreage of the plot from the following measurements :*

AB = 19·2 chains,	EF = 15 chains,
BC = 7 „	BG = 24 „
DE = 13 „	FD = 14 „

Shortest distance between GC *and* FD = 8 *chains.*

The plan of the plot is shewn in Fig. 35. By joining GB, GC, and FD, the plot is divided into three triangles GAB, GBC, FED and a trapezium GCDF.

Now the triangle GAB is right-angled at A so that its area $= \frac{1}{2}GA \cdot AB$. We must, therefore, find the length of GA. By the theorem of Pythagoras,

$$GA^2 = GB^2 - AB^2 = 24^2 - 19{\cdot}2^2 = (24 + 19{\cdot}2)(24 - 19{\cdot}2)$$
$$= 43{\cdot}2 \times 4{\cdot}8 = 9 \times 4{\cdot}8^2 ;$$

$$\therefore GA = 3 \times 4{\cdot}8 \text{ chains,}$$

and $$\triangle GAB = \tfrac{1}{2} \cdot 3 \times 4{\cdot}8 \times 19{\cdot}2 = 138{\cdot}24 \text{ sq. ch.}$$

The triangle GBC is also right-angled at B,

$$\therefore \ \triangle GBC = \tfrac{1}{2} \,.\, GB \,.\, BC = \tfrac{1}{2} \,.\, 24 \cdot 7 = 84 \text{ sq. ch.}$$

For the trapezium GCDF, we have by Art. 22,

$$\text{area} = \tfrac{1}{2}(GC + FD) \times 8 \text{ sq. ch.}$$

Now FD = 14 chains, and

$$GC = \sqrt{GB^2 + DC^2} = \sqrt{576 + 49} = \sqrt{625} = 25 \text{ chains};$$

$$\therefore \text{ area of trapezium} = \tfrac{1}{2} \,.\, (25 + 14) \,.\, 8 = 156 \text{ sq. ch.}$$

Finally, since the three sides of the triangle FDE are given, the rule of Prob. 6 gives, for the area, $\sqrt{21 \times 8 \times 7 \times 6} = 84$ sq. ch.

Hence the area of the whole plot

$$= 138 \cdot 24 + 84 + 156 + 84 \text{ sq. ch.}$$

$$= 462 \cdot 24 \text{ sq. ch.} = 46 \cdot 224 \text{ acres.}$$

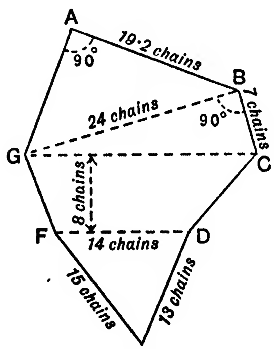


FIG. 35.—Plan of a plot of ground.

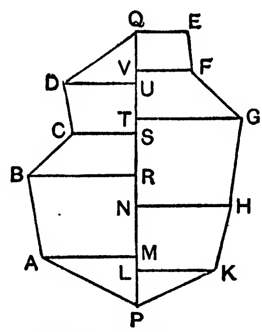


FIG. 36.—Field book plan.

25. Surveying. The Field Book. The measurement of land is called *surveying*, and in determining the area of an actual plot, the usual method is to map it out into right-angled triangles and trapeziums, where the boundaries are straight. Let ABC...P (Fig. 36) be the plan of a piece of level ground whose area is required. Two corners P, Q—called **stations**—are selected as far apart as possible, and a row of pegs driven into the ground to mark out the straight line PQ. This is called the **survey-line.** Stakes are next driven in at the chosen corners A, B, C, ... K, and by means of an instrument used for measuring angles, such as a

theodolite, offsets AM, BR, CS, ... KL are marked out perpendicular to PQ. These offsets are then measured by a surveying chain or an **offset staff**, *i.e.* a pole ten links long. The results are then recorded in a special way in a **field book**.

Suppose the measurements in links were as follows :

PL = 160, LK = 320, PM = 200, MA = 400,
PN = 432, NH = 400, PR = 560, RB = 464,
PS = 736, SC = 280, PT = 800, TG = 440,
PU = 960, UD = 320, PV = 1008, VF = 240,

PQ = 1200 and QE = 208 ;

these results would be entered in the book in the following manner :

Links

	to Q	
	1200	208 to E
	1008	240 to F
to D 320	960	
	800	440 to G
to C 280	736	
to B 464	560	
	432	400 to H
to A 400	200	
	160	320 to K
	From P	

It will be observed that this record is to be read **upwards**, the measurements in the middle column giving the distances from P of the feet of the offsets on the survey line, and the other columns giving the lengths of those offsets right or left. After the survey, the record is taken away, a plan prepared from it and the required area calculated, as in the following example.

Ex. 28. *Find the acreage of the piece of ground whose Field-book record is given above in Art.* 25.

In all cases where a record is given, a plan must first be prepared. This is best done on squared paper. In the present case, however, the plan is already shewn in Fig. 36, but it would be a useful exercise for the pupil to draw a new plan from the record given, without reference to Fig. 36.

From the diagram it is evident that the figure is divided into three right-angled triangles and seven trapeziums; hence, expressing all our measurements in chains, we have:

$$
\begin{array}{lrll}
\text{area of} & \mathsf{PLK} = \tfrac{1}{2} \,.\, 1{\cdot}6 \times 3{\cdot}2 & = 2{\cdot}56 & \text{square chains} \\
\text{,,} & \mathsf{LNHK} = \tfrac{1}{2} \,.\, 2{\cdot}72 \times 7{\cdot}2 & = 9{\cdot}792 & \text{,,} \\
\text{,,} & \mathsf{NTGH} = \tfrac{1}{2} \,.\, 3{\cdot}68 \times 8{\cdot}4 & = 15{\cdot}456 & \text{,,} \\
\text{,,} & \mathsf{TVGF} = \tfrac{1}{2} \,.\, 2{\cdot}08 \times 6{\cdot}8 & = 7{\cdot}072 & \text{,,} \\
\text{,,} & \mathsf{VQEF} = \tfrac{1}{2} \,.\, 1{\cdot}92 \times 4{\cdot}48 & = 4{\cdot}3008 & \text{,,} \\
\text{,,} & \mathsf{UQD} = \tfrac{1}{2} \,.\, 2{\cdot}4 \times 3{\cdot}2 & = 3{\cdot}84 & \text{,,} \\
\text{,,} & \mathsf{SUDC} = \tfrac{1}{2} \,.\, 2{\cdot}24 \times 6 & = 6{\cdot}72 & \text{,,} \\
\text{,,} & \mathsf{RSCB} = \tfrac{1}{2} \,.\, 1{\cdot}76 \times 7{\cdot}44 & = 6{\cdot}5472 & \text{,,} \\
\text{,,} & \mathsf{MRBA} = \tfrac{1}{2} \,.\, 3{\cdot}6 \times 8{\cdot}64 & = 15{\cdot}552 & \text{,,} \\
\text{,,} & \mathsf{PMA} = \tfrac{1}{2} \,.\, 2 \times 4 & = 4 & \text{,,} \\
& \therefore\ \text{Total area} & = 75{\cdot}8400 & \text{,,} \\
& & = 7{\cdot}584 & \text{or } \mathbf{7{\cdot}6} \text{ acres.}
\end{array}
$$

26. Curved Boundaries. When one of the boundaries is curved, the area may be found approximately by dividing the plan into narrow trapeziums by parallel offsets. In drawing the plan it is usual to take the survey line horizontal; the offsets are then vertical, and are often called **ordinates.**

Ex. 29. PQRS (Fig. 37) *is the plan of a plot of ground drawn to the scale of* 1 *cm. to* 20 *links. Find the area of the plot in sq. yd.*

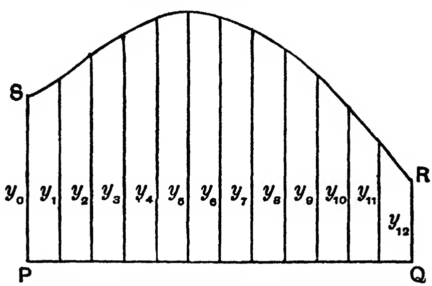


FIG. 37.—Plane area with a curved boundary.

It will be noticed that the boundary SR is curved. Suppose that on measuring PQ we find it is 12 cm. long. Divide PQ into 12 equal parts and draw the ordinates; let $y_0, y_1, y_2, \dots y_{12}$ be their respective lengths. The whole area is now divided approximately into 12 trapeziums, each 1 cm. wide.

It should be carefully observed that the ordinates must be sufficiently close together so that each segment of the curved boundary approximates to a straight line.

The area of the first is $\frac{1}{2}(y_0+y_1)$; and so on for the others.

Measure the lengths of the ordinates carefully, and record the results as follows:

Distance along PQ.	Measurements.	Sum of parallel sides.	Area.
0 cm.	y_0 = 5·2 cm.	—	
1 ,,	y_1 = 5·8 ,,	11·0 cm.	5·5 sq. cm.
2 ,,	y_2 = 6·6 ,,	12·4 ,,	6·2 ,,
3 ,,	y_3 = 7·2 ,,	13·8 ,,	6·9 ,,
4 ,,	y_4 = 7·8 ,,	15·0 ,,	7·5 ,,
5 ,,	y_5 = 8·0 ,,	15·8 ,,	7·9 ,,
6 ,,	y_6 = 7·8 ,,	15·8 ,,	7·9 ,,
7 ,,	y_7 = 7·4 ,,	15·2 ,,	7·6 ,,
8 ,,	y_8 = 6·8 ,,	14·2 ,,	7·1 ,,
9 ,,	y_9 = 6·0 ,,	12·8 ,,	6·4 ,,
10 ,,	y_{10} = 5·0 ,,	11·0 ,,	5·5 ,,
11 ,,	y_{11} = 3·8 ,,	8·8 ,,	4·4 ,,
12 ,,	y_{12} = 2·4 ,,	6·2 ,,	3·1 ,,

Total area = 76·0 sq. cm.

Now on our plan, 1 cm. = 10 links = 2·2 yd.

$$\therefore \ 1 \text{ sq. cm.} = 2{\cdot}2 \times 2{\cdot}2 = 4{\cdot}84 \text{ sq. yd.}$$

$$\therefore \ \text{Total area of plot} = 76 \times 4{\cdot}84 = \mathbf{367{\cdot}84} \text{ sq. yd.}$$

Problem 7. *To deduce a simple expression for the area of a figure having a curved boundary.*

From the above example, it will be seen that every ordinate between the end ones is used twice in finding the total area. Thus, suppose the width of each trapezium to be h; then from Fig. 37, the total area is

$$\tfrac{1}{2}h(y_0+y_1)+\tfrac{1}{2}h(y_1+y_2)+\ldots+\tfrac{1}{2}h(y_{11}+y_{12})$$
$$=h\{\tfrac{1}{2}(y_0+y_1)+y_1+y_2+\ldots+y_{11}\}.$$

In the same way, had there been n ordinates, the area would be

$$h\{\tfrac{1}{2}(y_0+y_n)+y_1+y_2+\ldots+y_{n-1}\}.$$

Hence the following rule:

Add half the sum of the first and last ordinates to the sum of the other ordinates, and multiply the result by the common distance between consecutive ordinates.

This is known as the **Trapezoidal Rule.**

27. Simpson's Rule. In order to obtain greater accuracy in finding the area of a plane figure with a curved boundary, Thomas Simpson, an English mathematician, devised the following rule in 1750. **Divide the base line into an even number (n) of equal parts ; let**

$$\mathbf{A} = \text{sum of end ordinates} = y_0 + y_n,$$
$$\mathbf{B} = \text{,, even ,,} = y_1 + y_3 + \ldots + y_{n-2},$$
and $$\mathbf{C} = \text{,, odd ,,} = y_2 + y_4 + \ldots + y_{n-1},$$
then $$\text{area} = \tfrac{1}{3}h(A + 4B + 2C),$$
where h = common distance between consecutive ordinates.

This is called **Simpson's Rule.**

Both the Trapezoidal Rule and Simpson's Rule may be applied to any figure having a curved boundary, whether the plan of a field or not. In determining the area of such a figure, it is convenient, however, to record the measurements in a manner similar to that adopted in the field book. The following example will clearly shew this :

Ex. 30. *Find* (i) *by the Trapezoidal Rule, and* (ii) *by Simpson's Rule, the area of* PQRS (Fig. 38) *from the following measurements, all given in links.*

Ordinates : $y_0=176$, $y_1=192$, $y_2=209$, $y_3=220$, $y_4=231$, $y_5=225$, $y_6=214$, $y_7=198$, $y_8=165$, $y_9=126$, $y_{10}=49$.

Length of base line = 100.

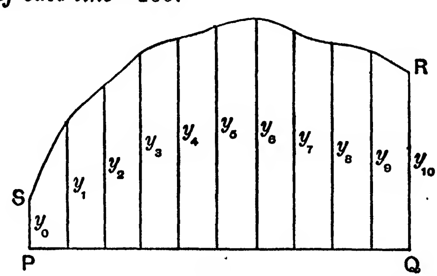


FIG. 38.

The base line is obviously divided into ten equal parts, each of which is, therefore, 10 links in length.

To sum the lengths of the ordinates, it is best to arrange the record as follows :

Links.

to Q.	End Ordinates.	Even Ordinates.	Odd Ordinates.
100	49	—	—
90	—	126	—
80	—	—	165
70	—	198	—
60	—	—	214
50	—	225	—
40	—	—	231
30	—	220	—
20	—	—	209
10	—	192	—
0	176	—	—
From P	—	—	—
Totals	225	961	819

Hence, by the Trapezoidal Rule,

$$\begin{aligned}\text{Area} &= 10\left(\tfrac{225}{2}+961+819\right)=18925 \text{ sq. links}\\ &= 1{\cdot}8925 \text{ sq. chains} = 0{\cdot}18925 \text{ acres}\\ &= \mathbf{0{\cdot}19} \text{ acre approximately.}\end{aligned}$$

By Simpson's Rule,

$$\begin{aligned}\text{Area} &= \tfrac{10}{3}(225+4\times 961+2\times 819)=19023 \text{ sq. links}\\ &= \mathbf{0{\cdot}19} \text{ acre approximately.}\end{aligned}$$

Note that Simpson's Rule gives a value nearer to 0·19 acre than that given by the Trapezoidal Rule.

EXERCISES 4c.

Find the area of each of the following quadrilaterals ABCD from the dimensions given :

1. AB = 25 in., BC = 8 in., CD = 15 in., DA = 26 in., $\angle$ BCD = 90°.

2. AB = 53 in., BC = 54·6 in., DA = 51 in., BD = 52 in., $\angle$ CBD = 90°.

3. AB = 37 ft., BC = 32 ft., DA = 13 ft., BD = 40 ft., $\angle$ BCD = 90°.

4. AB = 75 ft., BC = 41 ft., CD = 58 ft., DA = 78 ft., BD = 51 ft.

5. AB = 200 in., BC = 208 in., CD = 145 in., DA = 111 in., CA = 136 in.

6. AB = 77 cm., BC = 13 cm., BD = 85 cm., $\angle$A = $\angle$C = 90°.

7. AB = 56 cm., BD = 65 cm., CD = 16 cm., $\angle$A = $\angle$C = 90°.

8. BD = 14·5 yd., BC = 14·3 yd., DA = 1·7 yd., $\angle$A = $\angle$C = 90°.

9. ABCD represents a plot of ground; the perpendiculars from C and D to AB meet it in E and F respectively. Find the area of the plot

in acres from the following dimensions, all given in chains: AB = 86, AF = 24, FD = 17, EC = 23, EB = 44.

10. A plot is in the form of a five-sided figure ABCDE. Perpendiculars from E, D, C to AB meet it in F, G, H respectively. Find the acreage of the plot from the following dimensions, which are given in yards: AB = 268, AE = 37, FG = 125, GD = 20, FE = 12, HC = 45, BC = 53.

11. ABCD is a quadrilateral field in which the diagonals AC, BD are perpendicular to each other. AC = 134·9 metres and BD = 105 metres; find the area of the field in acres, taking 40·47 ares to the acre.

12. The diagonals AC, BD of a quadrilateral ABCD are perpendicular to each other, and $\angle$ BAD = 90°. AB = 120 yd., AD = 182 yd., and AC = 242 yd. Find the area of the quadrilateral in acres.

13. Find the acreage of a five-sided plot ABCDE, in which CD is parallel to BE, AB = 157 ft., CD = 145 ft., AE = 143 ft., BE = 140 ft., and the perpendicular distance between BE and CD = 88 ft.

For each of the following specimen entries in a Field-book, draw a plan on squared paper and calculate the acreage of the field to two places of decimals.

14. Links.

	602	
	387	215
	215	258
258	129	
215	43	
	0	

15. Links.

	767	
295	649	
295	472	236
	177	295
413	59	
	0	177

16. Links.

	848	
371	742	
	583	318
424	424	
265	159	318
	0	

17. Yards.

133	361	57
	190	76
114	114	
	95	114
152	38	
19	0	95

18. Links.

158	1106	
	948	316
395	790	
316	474	
	237	474
474	158	
	0	

19. Links.

	1455	
582	1261	
	1164	485
	582	679
388	388	
97	194	
	0	

20. Yards.

	629	
	592	259
74	555	
296	444	
	407	222
	185	259
222	148	
111	37	
	0	

21. Links.

435	1566	348
	1392	609
522	1044	
	870	348
696	609	
	522	522
261	261	
435	174	
	0	261

22. Yards.

	437	
161	345	
	322	69
115	276	
	207	184
184	138	
	115	115
69	69	
	23	92
	0	

23. Links.

	1168	
	949	438
438	730	
292	511	
	438	511
511	365	
	292	365
	146	146
365	73	
219	0	

24. Metres.

64·5	337	
	242	96
87·5	213	
	155	118
72·5	123	
	82	92
110	67	
	0	46

25. Metres.

8	322	
	303	104
102	172	
	159	55
106	109	
	51	96
118	28	
	0	

26. Links.

	1001	
210	945	
	845	238
266	721	
	602	399
476	308	
70	119	

27. Links.

	863	
	702	132
215	652	
	423	165
83	137	
	73	102

In each of the following examples the figure is bounded by a curved line, a straight base line, and the end ordinates. The area is divided into narrow strips of equal width by equidistant ordinates, and the lengths of these ordinates are given. Draw the figure on squared paper and calculate its area (i) by the Trapezoidal Rule and (ii) by Simpson's Rule.

28. Number of ordinates = 11; common interval between them = 2 mm. Lengths of ordinates in cm., 4, 3·96, 3·84, 3·64, 3·36, 3, 2·56, 2·04, 1·44, 0·76, 0.

29. Number of ordinates = 15; common interval between them = $\frac{1}{2}$ inch. Lengths of ordinates in inches, 5·2, 5·0, 4·8, 4·7, 4·6, 4·5, 4·4, 4·5, 4·6, 4·7, 4·8, 5·1, 5·3, 5·7, 6·2.

30. Number of ordinates = 19; common interval between them = 0·5 cm. Lengths of ordinates in cm., 4, 4·5, 4·9, 5·4, 5·8, 6·2, 6·4, 6·8, 7·2, 7·4, 7·7, 7·9, 8·2, 8·4, 8·5, 8·6, 8·7, 8·8, 8·9.

31. Number of ordinates = 17; common interval between them = 0·5 cm. Lengths of ordinates in cm.: 8, 8·80, 8·76, 8·72, 8·65, 8·57, 8·47, 8·34, 8·20, 8·03, 7·84, 7·63, 7·38, 7·10, 6·79, 6·44, 6·03.

32. Number of ordinates = 29; common interval = 2·5 cm. Lengths of ordinates in cm.: 0, 1·4, 2·8, 4, 5, 5·7, 6·2, 6·7, 7·2, 7·6, 8, 8·5, 8·9, 9·5, 10, 12·2, 15, 14·8, 14·5, 14·1, 13·7, 13·3, 12·8, 12·4, 11·7, 9·7, 7, 3·1, 0.

33. The depth of a river is measured in fathoms at intervals of 2 yards from bank to bank in a straight line with the following results: 0, 0·5, 1·2, 1·6, 3·1, 4·2, 5·6, 6·8, 7·4, 8·3, 8·4, 8·5, 8·8, 8·3, 8·2, 7·4, 6·5, 5·5, 3·8, 2·6, 1·5, 0·8, 0. Draw the section to a suitable scale, and calculate its approximate area in square yards by Simpson's Rule.

CHAPTER V

THE SINE AND COSINE OF AN ACUTE ANGLE

28. Solving a Triangle. Every triangle contains **six parts,** viz. three sides and three angles. When all these parts are known, the triangle is said to be **solved.** Generally when three parts are known, at least one of which must be a **side,** the remaining three parts may be found by calculation. We can test this fact in the case of a right-angled triangle.

Ex. 31. ABC *is a triangle in which* AB $=125$ *in.*, CA $=117$ *in.*, *and* C $=90°$; *solve the triangle.*

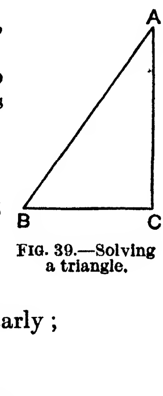


FIG. 39.—Solving a triangle.

Let ABC (Fig. 39) be the triangle; we have to find the length of BC and the sizes of the angles A and B.

By the theorem of Pythagoras we have :

$$BC^2 = AB^2 - CA^2 = 125^2 - 117^2$$
$$= (125+117)(125-117) = 242 \times 8 = 121 \times 16 = 44^2;$$
$$\therefore \ BC = 44 \text{ in.}$$

Now $\tan B = \frac{CA}{BC} = \frac{117}{44} = 2{\cdot}6591.$

From the tables, this is the tangent of 69° 24′ nearly ;

$$\therefore \ B = \mathbf{69° \ 24'}.$$

And since C $=90°$, therefore A + B $=90°$, and

$$A = 90° - B = 90° - 69° \ 24' = \mathbf{20° \ 36'}.$$

We can check this value by observing that

$$\tan A = \frac{BC}{CA} = \frac{44}{117} = 0{\cdot}3761.$$

Referring to the tables, this is the tangent of 20° 36′, thus shewing that our value is correct.

The six parts of the triangle are now known, hence it is completely solved.

Ex. 32. *Solve the triangle in which* $C=90°$, $B=31° 48'$, *and* $AB=81$ in.

Referring to Fig. 39, we have $A=90°-B=90°-31° 48'=\mathbf{58° 12'}$. To find BC and CA, we must proceed as follows:

$$CA/BC=\tan B=\tan 31° 48'=0{\cdot}62, \text{ from the tables};$$

$$\therefore\ CA=0{\cdot}62 \,.\, BC, \quad \text{and} \quad CA^2=(0{\cdot}62 \,.\, BC)^2=0{\cdot}3844 \,.\, BC^2.$$

But $$BC^2+CA^2=AB^2=81^2;$$

$$\therefore\ BC^2+0{\cdot}3844 \,.\, BC^2=81^2,$$

or $$1{\cdot}3844 \,.\, BC^2=81^2;$$

$$\therefore\ BC=81/\sqrt{1{\cdot}3844}=81/1{\cdot}176=\mathbf{68{\cdot}8} \text{ in.}$$

Finally, $$CA=0{\cdot}62 \,.\, BC=0{\cdot}62\times 68{\cdot}8=\mathbf{42{\cdot}7} \text{ in.}$$

The triangle is thus solved.

29. Two Important Ratios. In the latter example the triangle could be solved in a much shorter and neater way if we knew the relationship between CA and AB, and between BC and AB, just as we know the relationship between CA and BC when $\angle B$ is given. Since the ratio CA/BC, *i.e.* tan B, is constant, no matter how long the sides may be, provided B is constant, it is probable that the ratios CA/AB and BC/AB are also constant. Let us see whether this is so.

Ex. 33. *Draw any acute angle* XOQ; *take a series of points on* OQ *and drop perpendiculars to* OX. *Measure the sides of the right-angled triangles thus formed, and calculate the ratios of the values perpendicular/hypotenuse and base/hypotenuse for each. What do you infer from the results?*

The figure will be similar to Fig. 4 (p. 13). Make the required measurements and calculations, and tabulate as below:

Size of $\angle XOQ=53°$.

$P_1Q_1=6{\cdot}7$ cm.	$P_2Q_2=8{\cdot}7$ cm.	$P_3Q_3=10{\cdot}9$ cm.	$P_4Q_4=12{\cdot}7$ cm.	$P_5Q_5=13{\cdot}8$ cm.	$P_6Q_6=15{\cdot}4$ cm,
$OP_1=5{\cdot}0$ cm.	$OP_2=6{\cdot}5$ cm.	$OP_3=8{\cdot}2$ cm.	$OP_4=9{\cdot}6$ cm.	$OP_5=10{\cdot}4$ cm.	$OP_6=11{\cdot}7$ cm.
$OQ_1=8{\cdot}4$ cm.	$OQ_2=10{\cdot}9$ cm.	$OQ_3=13{\cdot}6$ cm.	$OQ_4=15{\cdot}9$ cm.	$OQ_5=17{\cdot}3$ cm.	$OQ_6=19{\cdot}4$ cm.
$\frac{P_1Q_1}{OQ_1}=0{\cdot}797$	$\frac{P_2Q_2}{OQ_2}=0{\cdot}798$	$\frac{P_3Q_3}{OQ_3}=0{\cdot}801$	$\frac{P_4Q_4}{OQ_4}=0{\cdot}798$	$\frac{P_5Q_5}{OQ_5}=0{\cdot}797$	$\frac{P_6Q_6}{OQ_6}=0{\cdot}794$
$\frac{OP_1}{OQ_1}=0{\cdot}594$	$\frac{OP_2}{OQ_2}=0{\cdot}596$	$\frac{OP_3}{OQ_3}=0{\cdot}603$	$\frac{OP_4}{OQ_4}=0{\cdot}604$	$\frac{OP_5}{OQ_5}=0{\cdot}601$	$\frac{OP_6}{OQ_6}=0{\cdot}603$

The ratios here are calculated to three places to shew that there is a slight variation, due to the measurements not being perfectly accurate. It is evident that the values in each of the last two lines do approach constant numbers, and taking the averages, we may say that for an angle of 53° the ratio of the perpendicular to the hypotenuse is approximately 0·798, whilst that of the base to the hypotenuse is approximately 0·600.

Ex. 34. *In a triangle* ABC *in which* $C=90°$, *the ratio* $CA/BC = 2{\cdot}06$. *Calculate the values of the ratios* CA/AB, *and* BC/AB. *What can be deduced from the results?*

Referring to Fig. 39 (p. 57), we have

$$AB^2 = BC^2 + CA^2.$$

But $CA/BC = 2{\cdot}06$, so that $CA = 2{\cdot}06 \,.\, BC$;

$$\therefore\ CA^2 = (2{\cdot}06 \,.\, BC)^2 = 4{\cdot}2436 \,.\, BC^2;$$

$$\therefore\ AB^2 = BC^2 + 4{\cdot}2436 \,.\, BC^2 = 5{\cdot}2436 \,.\, BC^2;$$

$$\therefore\ AB = \sqrt{5{\cdot}2436} \,.\, BC = 2{\cdot}29 \,.\, BC;$$

$$\therefore\ \frac{BC}{AB} = \frac{1}{2{\cdot}29} = \mathbf{0{\cdot}437},$$

and
$$\frac{CA}{AB} = \frac{2{\cdot}06 \,.\, BC}{AB} = 2{\cdot}06 \times 0{\cdot}437 = \mathbf{0{\cdot}900}.$$

Now, all we know about the triangle ABC is that CA is 2·06 times the length of BC, and this must be true however long or short the sides CA, BC may be. From this fact we have found that $BC = 0{\cdot}437 \,.\, AB$ and $CA = 0{\cdot}9 \,.\, AB$. It is evident, therefore, that these relations are true whatever the lengths of the sides may be consistent with the given conditions. The ratio CA/BC depends entirely upon the size of the angle B, as we have already seen; hence it follows that the ratios BC/AB, CA/AB, which depend upon the ratio CA /BC, are likewise dependent solely upon the size of the angle B.

EXERCISES 5A.

Repeat Ex. 33 in the case of each of the following angles:

1. 12°.	**2.** 21°.	**3.** 25°.	**4.** 30°.
5. 36°.	**6.** 43°.	**7.** 48°.	**8.** 57°.
9. 62° 30′.	**10.** 67° 48′.	**11.** 75°.	**12.** 82° 30′.

ABC is a triangle eight-angled at C; calculate the values of the ratios BC/AB, CA/AB, when the ratio CA/BC has each of the following values:

13. 1.	14. $\frac{1}{2}$.	15. $\frac{7}{24}$.	16. $\frac{120}{119}$.
17. 0·75.	18. 0·091.	19. 14·3.	20. 21·2.
21. k.	22. $\frac{1}{2}\left(\frac{m}{n}-\frac{n}{m}\right)$.		

23. A triangle ABC is right-angled at C, and CD is drawn perpendicular to AB. Shew that $\angle$ACD = B. If

(i) BC = 22·5, CA = 30, find the lengths of AD, DB and CD; hence calculate the ratios CA/AB, CD/BC, AD/CA and BC/AB, BD/BC, CD/CA. What do you infer from these results?

(ii) BC = a, CA = b, AB = c and CD = p, shew by finding the lengths of AD and BD that $p/a = b/c =$ AD$/b$, and $p/b = a/c =$ BD$/a$.

30. The Sine and Cosine of an Acute Angle. It will be evident from the results of the above exercises that for each angle, using the terms of Art. 11, the ratios perpendicular/hypotenuse, and base/hypotenuse, are constant, no matter how long the base, perpendicular and hypotenuse may be.

The ratio **perpendicular/hypotenuse** is called **the sine of the angle whose arms are the base and the hypotenuse,** and the ratio **base/hypotenuse is called the cosine** of the same angle.

Thus, referring to Fig. 39 (p. 57),

CA/AB = sine of B, which is written, **sin B,**

and **BC/AB = cosine of B,** which is written, **cos B.**

The results of Ex. 33 may now be written in the form,

$$\text{in } 53° = 0{\cdot}798 \text{ and } \cos 53° = 0{\cdot}600.$$

Just as in the case of the tangent, tables of sines and cosines have been constructed, by the methods of higher mathematics, for all the angles from 0° to 90°. If we refer to these tables we shall see that sin 53° = 0·7986, and cos 53° = 0·6018, so that our results, obtained in quite a rough way, are really quite near the tabulated values.

We shall now use these new ratios to re-solve the triangle in Ex. 32 (p. 58), and the simpler and neater method should be carefully observed.

Ex. 35. *Solve the triangle* ABC *in which* $C = 90°$, $B = 31° 48'$ *and* $AB = 81$ *in.*

As before, $A = 90° - B = \mathbf{58° \ 12'}$.

To find BC, we have $BC/AB = \cos B$,

i.e. $BC/81 = \cos 31° 48' = 0{\cdot}8499$,

from the tables;

$$\therefore \ BC = 81 \times 0{\cdot}8499 = \mathbf{68{\cdot}8} \text{ in.}$$

Also $CA/AB = \sin B$, *i.e.* $CA/81 = \sin 31° 48' = 0{\cdot}5270$;

$$\therefore \ CA = 81 \times 0{\cdot}5270 = \mathbf{42{\cdot}7} \text{ in.},$$

these results agreeing with those already found.

31. Construction of an Angle whose Sine or Cosine is given. From the definition of Art. 30, it will be seen that when the sine of an angle is given, we really know the ratio of the perpendicular to the hypotenuse of a right-angled triangle, and a geometrical construction of such a triangle is quite simple. Similarly, when the cosine is given, the ratio base/hypotenuse is known, and the triangle may be readily constructed. It must, however, always be remembered that the angle under consideration in such problems is that contained by the base and the hypotenuse.

Ex. 36. *Construct an angle whose sine is* 0·81; *measure the angle and calculate its cosine and tangent.*

Referring once again to Fig. 39 (p. 57), suppose B to be the angle, then

$$\sin B = CA/AB = 0{\cdot}81,$$

so that $CA = 0{\cdot}81 \times AB$.

Hence, whatever length we make AB, CA must be made equal to 0·81 of that length.

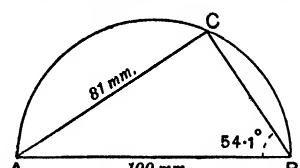


FIG. 40.—Construction of an angle.

Further, since C must be a right angle, the point C must lie on a semicircle whose diameter is AB, because we know from geometry that the angle in a semicircle is a right angle.

These facts, therefore, give the following simple construction.

Draw any straight line AB (Fig. 40), any convenient length. Suppose we make it 100 mm. long.

On AB describe a semicircle.

With centre A and radius $= 0\cdot81 \times AB = 0\cdot81 \times 100 = 81$ mm., describe an arc cutting the semicircle at C.

Join CA, CB, then $\angle ABC$ is the required angle.

For $\angle C$ is a right angle, and

$$\sin B = CA/AB = 81/100 = 0\cdot81$$

On measurement, the angle $B = \mathbf{54\cdot1^\circ}$.

To calculate the cosine and tangent of B, we must first find the length of BC.

From the theorem of Pythagoras, we have

$$BC^2 = AB^2 - CA^2 = 100^2 - 81^2 = 181 \times 19 = 3439,$$

$$\therefore\ BC = \sqrt{3439} = 58\cdot6\,;$$

$$\therefore\ \cos B = BC/AB = 58\cdot6/100 = \mathbf{0\cdot586}.$$

and $$\tan B = CA/BC = 81/58\cdot6 = \mathbf{1\cdot38}.$$

These values may be checked by the tables; thus we find $\cos 54\cdot1^\circ = 0\cdot5864$, and $\tan 54\cdot1^\circ = 1\cdot3814$; hence our results are approximately correct.

32. Complementary Angles. In any right-angled triangle ABC, as shewn in Fig. 39, we know that since $\angle C = 90^\circ$, therefore $A + B = 90^\circ$, *i.e.* the angles A and B are complementary. Now, from Ex. 9 (p. 16), we have seen that for two such angles,

$$\tan A \,.\, \tan B = 1.$$

Let us therefore investigate whether any similar relations exist between the sines and cosines of two complementary angles.

Ex. 37. *Write down the sines and cosines of each of the following angles and their complements:*

$$12^\circ,\ 22^\circ\,18',\ 34^\circ\,36',\ 47^\circ\,24',\ 67^\circ,\ 78^\circ\,42',\ 90^\circ.$$

Tabulate the results, and state what can be inferred from them.

Investigate with reference to a right-angled triangle, whether these inferences are true for any pair of complementary angles.

Angle	12°	22° 18′	34° 36′	47° 24′	67°	78° 42′	90°
Sine	0·2079	0·3795	0·5678	0·7361	0·9205	0·9806	1
Cosine	0·9781	0·9252	0·8231	0·6769	0·3907	0·1959	0
Complement	78°	67° 42′	55° 24′	42° 36′	23°	11° 18′	0°
Sine	0·9781	0·9252	0·8231	0·6769	0·3907	0·1959	0
Cosine	0·2079	0·3795	0·5678	0·7361	0·9205	0·9806	1

It is evident from this table that for each of the given angles, the sine is equal to the cosine of the complementary angle, and the cosine is equal to the sine of the complementary angle.

To see whether this is true for any pair of complementary angles, we have from Fig. 39, since $A+B=90°$, for the angle A,

$$\sin A = BC/AB, \quad \text{and} \quad \cos A = CA/AB,$$

and for the angle B,

$$\sin B = CA/AB, \quad \text{and} \quad \cos B = BC/AB,$$

i.e $\sin A = \cos B, \quad$ and $\quad \cos A = \sin B,$

so that the relations are true for all pairs of complementary angles.

Since $B=90° - A$, the relations may be expressed in the single form

$$\mathbf{\sin A = \cos (90° - A).}$$

33. The Cotangent. The relation just established is a very important one, but it is quite unlike that concerning the tangents given in Ex. 9. We can, however, express this latter relation in a precisely similar form to that connecting the sine and cosine of two complementary angles by introducing a new name for the ratio $1/\tan B$, or $1/\tan A$. These reciprocal ratios occur so often in problems that it is convenient to call them **cotangents** of the respective angles, thus

$1/\tan A =$ cotangent of $\angle A$, which is written $\mathbf{\cot A}$.

Similarly, $1/\tan B = \cot B$; and so on for any angle.

Now, according to the definitions of Art. 11, the tangent is the ratio, perpendicular/base, *i.e.* $\tan A = BC/C$ (Fig. 37).

Hence the cotangent is the ratio, base/perpendicular,

$$i.e. \quad \cot A = CA/BC.$$

We have, therefore, from Ex. 9, when $A+B=90°$,

$$\tan A \,.\, \tan B = 1,$$

so that $\quad \tan A = 1/\tan B = \cot B.$

But $B = 90° - A$;

$$\therefore \ \mathbf{\tan A = \cot (90° - A)},$$

a relation precisely like that found in Ex. 37, connecting the sine and cosine.

These important relations are sometimes utilised in constructing tables in a shortened form, as will be seen from the following skeleton:

Angle.	Sine.	Tangent.	Co-tangent.	Cosine.	
0°	0	0	∞	1	90°
1	·0175	·0175	57·2900	·9998	89
2	·0349	·0349	28·6363	·9994	88
3	·0523	·0524	19·0811	·9986	87
4	·0698	·0699	14·3007	·9976	86
...	...	...	...	...	...
...	...	...	...	...	...
41	·6561	·8693	1·1504	·7547	49
42	·6691	·9004	1·1106	·7431	48
43	·6820	·9325	1·0724	·7314	47
44	·6947	·9657	1·0355	·7193	46
45°	·7071	1·0000	1·0000	·7071	45°
	Cosine.	Co-tangent.	Tangent.	Sine.	Angle.

It will be observed that the angles at the ends of each horizontal line are complementary. The second column gives the sines from 0° to 45° and the cosines from 45° to 90°. The third column gives the tangents from 0° to 45° and the cotangents from 45° to 90°, and similarly for the fourth and fifth columns. Thus for the angles 0°-45°, the columns are named at the top, whilst for the angles 45°-90°, the columns are named at the bottom. It should be noted that such a table only gives the ratios for angles differing by one degree, the arrangement not being convenient for angles differing by less than a degree.

34. Two Important Relations. With respect to any angle A, the ratios sin A, cos A, tan A and cot A are generally called **Trigonometrical Functions**, and between these functions there are two very useful relations which we shall now derive.

Ex. 38. *For each of the following angles, write down the sine, cosine and tangent; then find for each the product of the cosine and the tangent. Tabulate the results, and state what may be inferred from them.*

$$18^\circ,\quad 29^\circ,\quad 38^\circ,\quad 43^\circ,\quad 54^\circ,\quad 60^\circ.$$

Denote any one of the angles by A, then for the first, $A = 18^\circ$, $\sin A = 0{\cdot}3090$, $\cos A = 0{\cdot}9511$, and $\tan A = 0{\cdot}3249$.

$\therefore\ \cos A \times \tan A = 0{\cdot}9511 \times 0{\cdot}3249 = 0{\cdot}3090$ to four places.

Repeating this for the other angles, we obtain the following table:

A	18°	29°	38°	43°	54°	60°
sin A	0·3090	0·4848	0·6157	0·6820	0·8090	0·8660
cos A	0·9511	0·8746	0·7880	0·7314	0·5878	0·5000
tan A	0·3249	0·5543	0·7813	0·9325	1·3764	1·7321
cos A . tan A	0·3090	0·4848	0·6157	0·6820	0·8091	0·8661

From these results we see that the product of cos A and tan A is practically equal to the value of sin A, the only variation being the small quantity 0·0001 in three cases. This is due to the fact that the tables are only approximately true to four places.

Ex. 39. *Find from the tables the sines and cosines of the following angles:* 13°, 30° 24′, 41° 18′, 54° 6′, 65° 6′, 80°; *then for each one calculate the values of the squares of the sine and cosine, and find their sum. Tabulate the results and state what inference may be drawn from them.*

Denote any one of the angles by A, then for the first, $A = 13^\circ$, and $\sin A = 0{\cdot}2250$, $\cos A = 0{\cdot}9744$.

Now the square of sin A may be written $(\sin A)^2$, but it is much more convenient to write it in the form $\sin^2 A$.

Similarly, the square of cos A is written $\cos^2 A$.

Note that the index is written after the name of the function, not after the angle, for it is the function, *i.e.* the *ratio* and not the *angle* which is to be squared.

Hence $\sin^2 A = \sin^2 13^\circ = 0{\cdot}2250^2 = 0{\cdot}0506$ to four places,

and $\qquad \cos^2 A = \cos^2 13^\circ = 0{\cdot}9744^2 = 0{\cdot}9495$;

$$\therefore\ \sin^2 A + \cos^2 A = 0{\cdot}0506 + 0{\cdot}9495 = 1{\cdot}0001.$$

Proceeding in this way with each of the remaining angles, we obtain the following table:

A	13°	30° 24′	41° 18′	54° 6′	65° 6′	80°
sin A	0·2250	0·5060	0·6600	0·8100	0·9070	0·9848
cos A	0·9744	0·8625	0·7513	0·5864	0·4210	0·1736
$\sin^2$ A	0 0506	0.2560	0·4356	0·6561	0·8227	0·9698
$\cos^2$ A	0·9495	0·7439	0·5645	0·3439	0·1772	0·0301
$\sin^2$ A + $\cos^2$ A	1·0001	0·9999	1·0001	1·0000	0·9999	0·9999

These results shew that the sum of the squares of the sine and cosine of each of the angles is very nearly equal to unity. The values are, indeed, so near that we may infer that the slight variation from 1, amounting only to 0·0001, is probably due, as in Ex. 38, to the tables being only approximately true to four places.

PROBLEM 8. *To find the relations* (i) *between sin θ, cos θ and tan θ, and* (ii) *between sin θ, and cos θ where θ is any acute angle.*

Let POQ (Fig. 41) be any acute angle whose size is $\theta°$. Take any point S on OP and draw ST perpendicular to OQ; then TOS is a right-angled triangle, so that

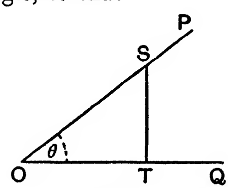


FIG. 41.

$$\sin\theta = \mathrm{TS}/\mathrm{OS}, \quad \cos\theta = \mathrm{OT}/\mathrm{OS}, \quad \text{and} \quad \tan\theta = \mathrm{TS}/\mathrm{OT}.$$

$$\therefore \ \mathrm{TS} = \mathrm{OS}\sin\theta, \text{ and } \mathrm{OT} = \mathrm{OS}\cos\theta,$$

$$\therefore \ \tan\theta = \frac{\mathrm{TS}}{\mathrm{OT}} = \frac{\mathrm{OS}\sin\theta}{\mathrm{OS}\cos\theta} = \frac{\sin\theta}{\cos\theta};$$

$$\therefore \ \tan\theta = \frac{\sin\theta}{\cos\theta},$$

no matter what value θ has, so long as it is an acute angle.

Again, $\mathrm{TS}^2 + \mathrm{OT}^2 = \mathrm{OS}^2.$

Divide out by OS^2, then

$$\left(\frac{TS}{OS}\right)^2+\left(\frac{OT}{OS}\right)^2=1,$$

i.e. $$(\sin\theta)^2+(\cos\theta)^2=1,$$

or $$\sin^2\theta+\cos^2\theta=1.$$

This is, therefore, the trigonometrical form of the theorem of Pythagoras.

35. Summary of Results. The trigonometrical facts that we have been considering so far are so important that a complete summary of them is here given for purposes of revision.

If $\theta°$ be any acute angle, then

(i) $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$,

(ii) $\cot\theta=\dfrac{1}{\tan\theta}=\dfrac{\cos\theta}{\sin\theta}$,

(iii) $\sin\theta=\cos(90°-\theta)$ and $\cos\theta=\sin(90°-\theta)$,

(iv) $\tan\theta=\cot(90°-\theta)$ and $\cot\theta=\tan(90°-\theta)$,

(v) $\sin^2\theta+\cos^2\theta=1$.

EXERCISES 5B.

Construct the angle whose sine is given in each of the following examples. Measure the angle and find its cosine and tangent without using the tables.

1. $\frac{24}{25}$. 2. $\frac{4}{5}$. 3. 0·5.

4. 0·81. 5. 0·91. 6. 0·907.

7. Construct an acute angle whose sine is $\frac{5}{11}$, and measure it to the nearest degree. (C.S.)

8. Given that $\sin A=\frac{15}{28}$, find A to the nearest minute. (N.U.T.)

9. Construct an angle whose sine is 0·45 and obtain by measurement from your figure the cosine and tangent of the angle. (U.L.C.I.)

10. Construct an angle whose sine is equal to its cosine, and measure it.

Construct an angle whose cosine is given in each of the following examples. Measure the angle and find by measurement its sine and tangent:

11. 0·8. 12. $\frac{44}{125}$. 13. 0·43.

14. $\frac{24}{25}$. 15. 0·51. 16. 0·274.

17. Find, from the tables, the angles whose sines are: 0·7986, 0·866, 0·9205, and find their sum.

18. Find, from the tables, the angles whose cosines are: 0·6018, 0·5, 0·3907, and find their sum.

19. If $\sin A = 0{\cdot}4003$, find A and $\cos A$; hence find the angle whose sine is equal to twice the product of $\sin A$ and $\cos A$. What relation does this angle bear to A?

20. If $\cos A = 0{\cdot}9003$, find A and $\sin A$; hence find B so that

$$\sin B = 2 \sin A \, . \cos A.$$

What relation is there between A and B?

21. Find the values of $\sin B \div (\sin A \, . \cos A)$ when $A = 5°$, $12°$, $21°$, $35°$, $41°$, and $B = 2A$. What inference can be drawn from the results?

22. The cosine of an angle is equal to $\cos 40° + \cos 80°$; find the angle.

23. The sine of an angle is equal to $\sin 10° \, 36' + \sin 11° \, 36' + \sin 30° \, 30'$; find the angle.

24. The cosine of an angle is equal to $\cos 55° + \sin 65° - \tan 55° \, 42'$; find the angle, and its complement.

25. If $\sin A = \frac{1}{4}(\sqrt{6} - \sqrt{2})$, find A and $\cos A$.

26. Given that $\cos A = \frac{1}{4}(\sqrt{5} + 1)$, find A, $\sin A$ and $\tan A$.

27. The sine of an angle is equal to $\frac{39}{89}$; find, without using the tables, the values of the cosine and tangent of the angle.

28. If $\cos A = \frac{13}{85}$, find $\sin A$ and $\tan A$ without using tables.

29. The tangent of an acute angle is $\frac{95}{168}$; find the sine and cosine of the angle.

30. Draw an angle of 35° and find from your figure the values of the sine, cosine and tangent of 35°. Use your results to find the value of $\sin^2 35° + \cos^2 35°$. (N.U.T.)

31. Find the values of $\cos^2 A(1 + \tan^2 A)$ when $A = 8°$, $15°$, $23°$, $35°$ and $40°$. What inference can be drawn from the results?

32. If A denote the angle whose tangent is $\frac{60}{11}$, verify that

$$\cos^2 A(1 + \tan^2 A) = 1,$$

and prove this relation is true for any acute angle. (D.S.)

33. The cosine of an angle is equal to $\cos^2 A - \sin^2 A$; find the angle for each of the following values of A: $11°$, $16°$, $24°$, $32°$, $41°$. What relation does this angle bear to A?

34. If $\cos B = 2 \cos^2 A - 1$, find the values of B when $A = 6°$, $12°$, $25°$, $31°$, $42°$. What relation is there between B and A?

35. Given that $\cos B = 1 - 2 \sin^2 A$, find B for the following values of A: $8°$, $15°$, $20° \, 30'$, $33°$, $40° \, 42'$. What relation do the results reveal between A and B?

Solve each of the following triangles in which C is a right angle :

36. $a=11{\cdot}7$, $c=12{\cdot}5$.
37. $b=14{\cdot}3$, $c=14{\cdot}5$.
38. $a=35$, $b=12$.
39. $a=13{\cdot}2$, $b=8{\cdot}5$.
40. $a=21{\cdot}6$, $b=8{\cdot}3$.
41. $c=32$, $A=48°\ 42'$.
42. $c=54{\cdot}7$, $A=73°\ 18'$.
43. $a=29{\cdot}31$, $A=77°\ 42'$.
44. $a=45$, $A=13°$.
45. $c=52$, $B=71°\ 48'$.
46. $c=85{\cdot}2$, $B=65°\ 6'$.
47. $a=13{\cdot}5$, $B=77°$.
48. $a=45{\cdot}5$, $B=24°\ 30'$.
49. $b=71$, $A=73°\ 30'$.
50. $b=39{\cdot}7$, $A=7°$.
51. $b=36{\cdot}9$, $B=67°\ 18'$.
52. $b=49{\cdot}5$, $B=41°\ 18'$.
53. $a+b=161$, $a-b=73$.
54. $a+b+c=40$, $c-a=2$.

55. Give a geometrical construction for drawing a triangle ABC in which $AC=10$ cm., angle $C=90°$, and $\sin A=0{\cdot}65$.

If CB is produced to D so that $BD=2\,.\,CB$, calculate the angle CAD. (C.S.)

56. ABC is a triangle in which $A=70°$, $B=90°$ and $AC=10$ cm. Find the length of AB and BC.

If BC is produced to D so that $CD=4$ cm., calculate the angle BAD and the length of AD. (C.S.)

57. A ladder 30 feet long placed against a vertical wall at an inclination of 56° to the ground reaches to a point which is 4 feet below the top of the wall. Calculate the inclination at which the ladder must be placed against the wall in order that it may just reach to the top of the wall. (C.S.)

58. A ladder 25 ft. long rests against the vertical wall of a house with its foot 7 ft. 9 in. from the foot of the wall. Find the angle which the ladder makes with the vertical. If the ladder slips 2 ft. down the wall, what is now the angle with the vertical ? (N.U.T.)

59. A lighthouse is 31° East of North from a harbour 5 miles away. A ship leaves the harbour in a direction 10° West of North. How far has the ship sailed when it is nearest to the lighthouse, and how much further when the lighthouse bears due East from the ship ? (J.M.B.)

60. The corner A of a square ABCD is joined to E, the middle point of BC. Find, correct to four significant figures, the sine, cosine and tangent of the angle BAE, and the magnitude of the angle AED. (D.S.)

CHAPTER VI

AREAS OF RECTILINEAL FIGURES IN TERMS OF TRIGONOMETRICAL FUNCTIONS

36. Trigonometrical Functions as Conversion Factors. Let ABC be any triangle in which C is a right angle (see Fig. 39, p. 57); then, in the notation of Art. 22, $\sin A = BC/AB = a/c$; $\cos A = CA/AB = b/c$; $\tan A = BC/CA = a/b$, and $\cot A = 1/\tan A = CA/BC = b/a$. Clearing each of these relations of fractions, we have:

$$a = c \sin A, \quad b = c \cos A, \quad a = b \tan A, \text{ and } b = a \cot A.$$

Hence, if the angle A and the hypotenuse c be known,

c multiplied by $\sin A$ gives the length of the perpendicular a, and

c multiplied by $\cos A$ gives the length of the base b;

or if $\angle A$ and one of the sides a or b are known,

b multiplied by $\tan A$ gives the length of the perpendicular a, whilst

a multiplied by $\cot A$ gives the length of the base b.

We see, therefore, that the functions $\sin A$, $\cos A$, $\tan A$, and $\cot A$ may be regarded as **multipliers** or **conversion factors**, which will convert one side of a right-angled triangle into another side. In the same way $\sin B$, $\cos B$, $\tan B$, and $\cot B$ may also be used as conversion factors.

Ex. 40. *Find the sides a, b of a triangle* ABC *in which* $C = 90°$, $B = 52°$ *and* $c = 43{\cdot}6$ *in.*

Here the angle B is given, so that a must be taken as the base, and b as the perpendicular.

Hence $\sin B = b/c$, or $b = c \sin B = 43{\cdot}6 \sin 52° = 43{\cdot}6 \times 0{\cdot}7880$
$= \mathbf{34{\cdot}4}$ in.,

and $\cos B = a/c$, or $a = c \cos B = 43{\cdot}6 \cos 52° = 43{\cdot}6 \times 0{\cdot}6157$
$= \mathbf{26{\cdot}8}$ in.

37. Area of a Triangle. We know from Problem 3 that the area of a triangle is equal to half the product of the base and the altitude. We have also just seen how the perpendicular of a right-angled triangle may be found when the hypotenuse and one angle are known. Let us, therefore, apply these facts to find the area of any triangle when two sides and the included angle are known.

Ex. 41. ABC *is a triangular plot of land in which* CA $=979$ *yards,* BC $=832$ *yards and* $\angle$ ACB $=53°$; *find the area of the plot in acres.*

Let ABC (Fig. 42) represent the plot; draw AD perpendicular to BC; then the area of the triangle

$$=\tfrac{1}{2}BC \,.\, AD = 416 \,.\, AD \text{ sq. yd.}$$

But since ACD is a right-angled triangle in which $\angle C$ and CA are known, we have

$$AD = CA \sin C = 979 \sin 53° = 979 \times 0{\cdot}7986.$$

$\therefore$ area of triangle

$$= 416 \times 979 \times 0{\cdot}7986 \text{ sq. yd.}$$

$$= \frac{416 \times 979 \times 0{\cdot}7986}{4840} = 69{\cdot}8 \text{ acres.}$$

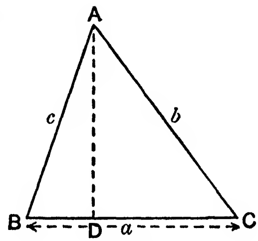


FIG. 42.

38. General Rule. It is quite easy to find a general expression for the area of any triangle in terms of two sides and the included angle.

PROBLEM 9. *To find the area of a triangle when two sides and the included angle are given.*

Since the sum of the three angles of every triangle is 180°, two, at least, of the angles must be acute. We shall therefore suppose at present that the given angle is acute.

Referring to Fig. 42, let the sides $BC=a$, $AB=c$, and the acute angle B be given; then the area of the triangle $=\tfrac{1}{2}a \,.\, AD$, where D is the foot of the perpendicular from A to BC.

But from the right-angled triangle ABD, $AD = c \sin B$,

$$\therefore \text{ area of triangle} = \tfrac{1}{2}ac \sin B.$$

In a similar manner it may be shewn that when C is acute, the area is $\tfrac{1}{2}ab \sin C$, and when A is acute, the area is $\tfrac{1}{2}bc \sin A$; hence

the area of a triangle is measured by

$\tfrac{1}{2}$. (product of two sides) × (sine of angle between them).

It will be seen later that this rule is also true when the given angle is not acute. See Art. 62 (p. 107).

39. Area of a Triangle when Two Angles and a Side are given. By the use of trigonometrical functions, the area of a triangle may also be found when one side and two angles are known. The given side may be a common arm of the given angle as in Ex. 42, or it may be opposite one of them as in Ex. 43.

Ex. 42. *Find the area of a triangle* ABC *in which* $a = 98{\cdot}1$ *in.,* $B = 57^\circ\ 42'$ *and* $C = 65^\circ\ 24'$.

Referring to Fig. 42, we have $BD = AD \cot B$, and $DC = AD \cot C$,

$$\therefore\ a = BC = BD + DC = AD\,(\cot B + \cot C)$$
$$= AD\,(\cot 57^\circ\ 42' + \cot 65^\circ\ 24')\,;$$
$$\therefore\ 98{\cdot}1 = AD(0{\cdot}6322 + 0{\cdot}4578) = AD \times 1{\cdot}09,$$
$$\therefore\ AD = 98{\cdot}1/1{\cdot}09 = 90 \text{ in.},$$
$$\therefore\ \Delta = \tfrac{1}{2}\,.\,a\,.\,AD = \tfrac{1}{2}\,.\,98{\cdot}1 \times 90 = \mathbf{4414{\cdot}5} \text{ sq. in.}$$

Note that if in finding the value of $\cot B + \cot C$, no table of cotangents is available, relation (iv) of Art. 35 (p. 67) may be used; thus

$$\cot 57^\circ\ 42' = \tan\,(90^\circ - 57^\circ\ 42') = \tan 32^\circ\ 18' = 0{\cdot}6322.$$

Similarly, $\cot 65^\circ\ 24' = \tan 24^\circ\ 36' = 0{\cdot}4578.$

Ex. 43. *A triangle* ABC *has* $b = 38$ *in. and* $B = 54^\circ\ 12'$; *calculate its area* (i) *when* $C = 71^\circ\ 48'$, *and* (ii) *when* $C = 23^\circ\ 30'$.

(i) In this case the given side is not a common arm to the given angles, but since B and C are known, we can easily find A and then proceed as in Ex. 42. Thus

$$A = 180^\circ - (54^\circ\ 12' + 71^\circ\ 48') = 180^\circ - 126^\circ = 54^\circ.$$

If, therefore, p be the length of the perpendicular from B to AC, $p\,(\cot C + \cot A) = b$, or putting in the values, $1{\cdot}0553p = 38$, so that

$$p = 38/1{\cdot}0553\,;$$

$$\therefore\ \Delta = \tfrac{1}{2}pb = \frac{38 \times 38}{2 \times 1{\cdot}0553} = \mathbf{684{\cdot}2} \text{ sq. in.}$$

(ii) Here $A = 180^\circ - (54^\circ\ 12' + 2^\circ\ 30') = 102^\circ\ 18'$, hence the perpendicular from B to CA falls outside the triangle. Let it meet CA produced in E, then putting $BE = p$,

$$AE = p \cot BAE = p \cot\,(180^\circ - A) = p \cot 77^\circ\ 42'\,;$$

$$\therefore\ b = \mathrm{CE} - \mathrm{AE} = p(\cot \mathrm{C} - \cot 77° \, 42'),$$

i.e. $$38 = p(2{\cdot}2998 - 0{\cdot}2180) = 2{\cdot}0818p,$$

$$\therefore\ p = 38/2{\cdot}0818,$$

and $$\Delta = \tfrac{1}{2}pb = 38 \times 38/4{\cdot}1636 = \mathbf{346{\cdot}8} \text{ sq. in.}$$

40. Area of a Parallelogram. The above methods of finding the area of a triangle may be applied to the parallelogram, or, indeed, any quadrilateral, when sufficient measurements are given.

Ex. 44. ABCD *is a parallelogram in which*

$$\mathrm{AB} = 52 \textit{ in.}, \quad \mathrm{BC} = 25 \textit{ in.} \quad \textit{and} \quad \angle \mathrm{ABC} = 73° \, 44';$$

find its area. In BC *produced a point* F *is to be taken so that the area of* ABF *is* 0·8 *of the area of the parallelogram. Find the length of* CF.

Let ABCD (Fig. 43) be the parallelogram; draw CE perpendicular to AB, then

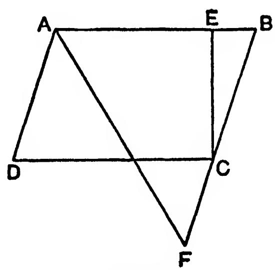


FIG. 43.

$$\mathrm{CE} = \mathrm{BC} \sin \mathrm{EBC} = 25 \sin 73° \, 44' = 25 \times 0{\cdot}96 = 24 \text{ in.}$$

$$\therefore \text{ by Prob. 2, area of parallelogram} = \mathrm{AB} \,.\, \mathrm{EC} = 52 \times 24$$
$$\mathbf{1248} \text{ sq. in.}$$

Turning to the second part of the question, let F be the required point in BC produced; then $\Delta \mathrm{ABF} = 0{\cdot}8 \times 1248$ sq. in.

But by Prob. 9, $\Delta \mathrm{ABF} = \tfrac{1}{2} \,.\, \mathrm{AB} \,.\, \mathrm{BF} \,.\, \sin \mathrm{ABF} = 26 \,.\, \mathrm{BF} \,.\, 0{\cdot}96$;

$$\therefore\ 26 \,.\, \mathrm{BF} \,.\, 0{\cdot}96 = 0{\cdot}8 \times 1248;$$

$$\therefore\ \mathrm{BF} = \frac{0{\cdot}8 \times 1248}{26 \times 0{\cdot}96} = 40 \text{ in.},$$

i.e. $\mathrm{BC} + \mathrm{CF} = 40$, so that $\mathrm{CF} = 40 - \mathrm{BC} = 40 - 25 = \mathbf{15}$ **in.**

41. Area of a Regular Polygon. A polygon is a plane rectilineal figure having more than *four sides*. It is *regular* when all its sides are equal.

When there are **five sides**, the polygon is called **a pentagon.**

,,	,,	**six**	,,	,,	,,	**a hexagon,**
,,	,,	**seven**	,,	,,	,,	**a heptagon,**
,,	,,	**eight**	,,	,,	,,	**an octagon,**
,,	,,	**nine**	,,	,,	,,	**a nonagon,**
,,	,,	**ten**	,,	,,	,,	**a decagon,**

and generally when there are n sides, n being any whole number, the polygon is called an **n-gon.** The area of a polygon may be found by dividing it into as many triangles as there are sides. The following example should be carefully studied.

Ex. 45. *Find the area of a regular octagon each of whose sides is* 5 *cm. in length.*

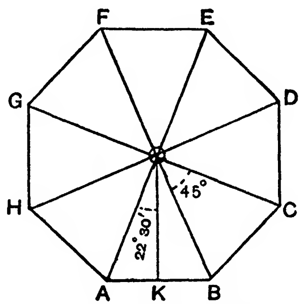


FIG. 44.—Area of a regular octagon.

Let ABCDEFGH (Fig. 44) be the octagon, and O its centre. Since all its sides are equal, by joining each angular point to O, we have eight equal isosceles triangles, of which AOB is one.

Now $\angle AOB = \frac{1}{8}$ of total angle at $O = \frac{1}{8} \,.\, 360° = 45°$.

Draw OK perpendicular to AB; then since AOB is isosceles, $AK = KB = \frac{1}{2}AB = 2{\cdot}5$ cm., and $\angle AOK = \angle KOB = \frac{1}{2}\angle AOB = 22° \, 30'$

Again, $OK = AK \cot AOK = 2{\cdot}5 \cot 22° \, 30' = 2{\cdot}5 \times 2{\cdot}4142$ cm.;

$$\therefore \; \triangle AOB = \tfrac{1}{2} \,.\, AB \,.\, OK = 2{\cdot}5 \times 2{\cdot}5 \times 2{\cdot}4142 \text{ sq. cm.}$$

And since the octagon contains 8 triangles each equal to AOB,

$$\therefore \; \text{area of octagon} = 8 \times 2{\cdot}5 \times 2{\cdot}5 \times 2{\cdot}4142 \text{ sq. cm.}$$
$$= \mathbf{120{\cdot}7} \text{ sq. cm.}$$

EXERCISES 6.

ABC is a triangle right-angled at C; find

1. a and b, when $c=52$ in. and $A=30°$.
2. a and b, when $c=75$ in. and $B=63°$.
3. b and c, when $a=200$ cm. and $B=45°$.
4. a and c, when $b=237{\cdot}3$ cm. and $A=55°\ 36'$.
5. b and c, when $a=38$ ft. and $A=71°\ 48'$.
6. c and a, when $b=71$ ft. and $A=73°\ 30'$.

7. A straight railway runs for a quarter of a mile up a slope inclined at 20° to the horizontal. Through what vertical height in feet does the railway ascend?

8. A ladder inclined at 66° to the horizontal reaches a point on a vertical wall 51 ft. from the ground. What is the length of the ladder, and how far is its foot from the wall?

9. ABC is a triangular plot of ground in which $\angle C=90°$, $\angle B=36°\ 18'$, and $AB=37$ chains. Find the lengths of CA and BC.

10. To find the height of a factory chimney, a person observes that the angle of elevation of the top is 53° at a point on the ground. On walking 53 ft. in a straight direction towards the chimney he finds the angle of elevation of the top is now 61°. Calculate the height of the chimney.

11. Part of a jib-crane is represented by a triangle ABC, in which AB is the vertical post 20 ft. high, A being on the ground, BC is the tie-rope, and CA is the jib, which is 41 ft. long. When the angle BAC is 38°, calculate (i) the height of C above the ground, and (ii) the length of the tie-rope BC.

12. A rectangular table, sides 5 ft. and 7 ft., is placed with one of its shorter sides across a corner of a room touching the two walls and making an angle of 40° with one of them. Find the distances of the other two corners of the table from the two walls. (J.M.B.)

13. A pendulum 6 ft. 3 in. long swings through a total angle of 12° 24′. Find the horizontal distance between the two extreme positions of the bob, and the distance the bob is below this line when the pendulum is in the vertical position.

14. The top of a hill is reached by a path which runs for 200 yards along a slope of 20° to the horizontal, then for 300 yards along a slope of 15° and then for 400 yards along a slope of 10°. Calculate the height of the hill to the nearest foot. (C.S.)

15. To find the distance of an inaccessible object C from a station A, a base line AB, 95 yards long, is measured, and the angles BAC, ABC are then observed to be 46° and 61° respectively. Calculate the distance AC in yards.

Find the area of each of the following triangles in the specified units :

16. $a=6$ yd., $b=7$ yd., $C=58°$, in sq. yd.
17. $b=18$ ft., $c=21$ ft., $A=65° 30'$, in sq. yd.
18. $c=375$ chains, $a=352$ chains, $B=35° 6'$, in acres.
19. $a=75$ chains, $b=72$ chains, $C=81° 54'$, in acres.
20. $b=176$ yd., $c=165$ yd., $A=56° 6'$, in acres.
21. $a=862$ yd., $b=575$ yd., $C=75° 28'$, in acres.
22. $c=768$ m., $a=625$ m., $B=83°$, in hectares.
23. $a=432$ m., $b=325$ m., $C=71° 48'$, in hectares.
24. $b=888$ m., $c=875$ m., $A=7° 45'$, in acres, taking 40·47 ares to an acre.
25. $c=87{\cdot}6$ m., $a=300$ m., $B=42° 25'$, in acres, using the relation given in Ex. 24.
26. $a=14$ in., $B=81° 12'$, $C=61° 24'$, in sq. in.
27. $b=67$ yd., $C=51°$, $A=39° 48'$, in sq. yd.
28. $c=28$ chains, $A=73° 18'$, $B=68° 12'$, in acres.
29. $a=6{\cdot}3$ m., $B=63° 54'$, $C=52° 24'$, in sq. m.
30. $b=35{\cdot}8$ m., $C=46°$, $A=50° 30'$, in ares.
31. $a=2$ yd. 2 ft., $A=69° 18'$, $B=30°$, in sq. ft.
32. $b=20$ chains, $A=54° 6'$, $B=39° 12'$, in acres.
33. $c=12{\cdot}5$ m., $B=65° 30'$, $C=50° 30'$, in sq. m.
34. $a=157$ m., $A=58°$, $B=21° 6'$, in hectares.
35. $b=134{\cdot}9$ m., $B=48° 36'$, $C=75° 55'$, in acres, using the relation given in Ex. 24.

36. A triangular area ABC is represented on a map, the scale of which is 2 inches to the mile, by a triangle A′B′C′. $A'B'=\frac{1}{2}$ in., $A'C'=\frac{3}{4}$ in., $\angle A=30°$. Find the number of square yards in the area ABC. (J.M.B.)

37. Find the area in square feet of a parallelogram ABCD in which AB = 2 ft. 1 in., BC = 1 ft. 4 in., and $\angle ABC=54° 6'$.

38. ABCD is a parallelogram in which AB = 35 ft., BC = 18 ft., and $\angle BAD=37° 18'$; find its area in square yards.

39. The area of a parallelogram ABCD is 709·5 square yards; CD = 25 yards, and $\angle BAD=41° 18'$. Find the length of AB and the angle ABC.

40. The diagonals AC, BD of a parallelogram ABCD intersect at E. AC = 74 ft., BD = 102 ft. and $\angle AED=21°$. Find the area of the parallelogram to the nearest square foot.

41. The diagonals AC, BD of a parallelogram ABCD intersect at E. AC = 58 yd., BD = 74 yd. and $\angle AED=44°$. Find the area of the parallelogram to the nearest square yard.

42. ABCD is a parallelogram in which the lengths of the sides AB, BC are 12 in. and 8 in. respectively, and the angle A is 65°. Calculate the

lengths of the sides and the diagonals of the rectangle formed by the bisectors of the angles of the parallelogram. (C.S.)

43. In a quadrilateral field ABCD, the side AB is 55 yards, AC is 65 yards and AD is 39 yards. Angle BAC is 25° and ADC is 90°. Find the area of the field. (J.M.B.)

44. ABCD is a trapezium whose sides AB, CD are parallel to one another. The length of AB is 12 in., the length of AD is 21 in. The angle ADC = 36° 20′, the angle BCD = 63° 40′. Find the area of ABCD. (J.M.B.)

45. The plan of a hall floor is a hexagon ABCDEF in which AB = FE, BC = DE, ACDF is a rectangle, and BE is parallel to CD. Calculate the area of the floor in sq. yd. from the following dimensions: AC = 92 ft., AF = 47 ft., ∠ BAC = ∠ DFE = 20°, and ∠ BCA = ∠ EDF = 47°.

46. Find the area of a regular hexagon each of whose sides is 2·5 in.

47. A piece of wire 24 inches long is bent into the form of a regular pentagon. Find the area of the pentagon. (L.M.)

48. A plot of ground in the shape of a regular octagon has each of its sides 229·6 metres in length. Find its area in hectares.

49. In a triangle ABC, $b = 558$ ft., $c = 335$ ft., and A = 32° 24′. By equating two expressions for the area of the triangle, find the length of the bisector of the angle A, between A and BC.

50. The bisector AD of the angle A of a triangle ABC, meets BC in D. AB = 171 ft., A = 40° and AD = CA. Find the length of CA.

51. By writing down the area of an acute-angled triangle ABC in three forms and equating them two at a time, shew that

$$\sin A/a = \sin B/b = \sin C/c;$$

hence prove that the area may be expressed in the form

$$\tfrac{1}{2}a^2 \sin B \sin C/\sin A.$$

Use this result to calculate the area when $a = 9$ yards, A = 21° 6′, and B = 81° 54′.

52. If the angles B, C, and the side a of a triangle ABC are given, shew that its area is $\dfrac{a^2}{2(\cot B + \cot C)}$; hence find a when B = 61°, C = 65° 30′ and the area is 1262·5 sq. cm.

53. If the angles B, C and the side b of a triangle ABC are given, shew that its area is $\frac{1}{2}b^2 \sin^2 C(\cot B + \cot C)$; hence find b when B = 69° 18′, C = 30°, and the area = 16·88.

54. ABCD is a trapezium in which AB and CD are parallel. If AB = a, CD = b, AD = c, and ∠ ADC = θ, prove that its area is $\frac{1}{2}c(a+b)\sin\theta$.

Calculate the area in acres when $a = 289$ yd., $b = 197$ yd., $c = 45$ yd., and $\theta = 58° 36'$.

55. The bisector of the angle A of a triangle ABC meets BC in P; if AP = t, shew that the area of the triangle is $\frac{1}{2}t(b+c)\sin\frac{1}{2}A$.

Hence find t when $b = 45$, $c = 36$ and A = 71° 48′.

CHAPTER VII

THE CIRCLE

42. The Circle. If we take an ordinary pair of compasses and open the legs at any angle, and then, placing the needle point at a point O (Fig. 45), revolve the pencil point about O, taking care to keep the legs open at the same angle, a closed curve ARBS, called a **circle** is described. The curved boundary ARBS is known as the **circumference**, and the fixed point O as the **centre**. Since every point on the circumference is the same distance from the centre, we may say that a **circle is the locus of a point which moves in such a way that its distance from a fixed point is always the same.**

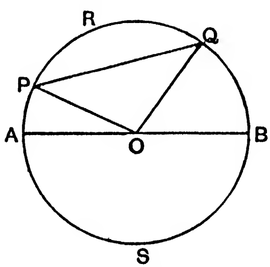


FIG. 45.—The circle.

The constant distance OA, OP, OQ, or OB between any point on the circumference and the centre is called the **radius** of the circle, and any straight line such as PQ or AB joining two points on the circumference is called a **chord**. When a chord like AB also passes through the centre, it is called a **diameter**. It will thus be clear that the length of a diameter of a circle is just twice the length of its radius.

Ex. 46. *Find an approximate relation between the length of the circumference of a circle and its diameter.*

Take several circular solids and measure carefully their diameters and circumferences in centimetres. The diameter may be measured by placing the solid between two blocks with the

ends cut square to the surface, and then measuring the distance between the blocks. The circumference may also be found by wrapping a thin strip of paper evenly round the circular edge of the solid until the paper begins to overlap; then prick through the double thickness, unwrap the paper and measure the distance between the two pin-holes.

Having made all the measurements, divide the length of the circumference of each circle by its diameter. The following results were obtained by a pupil.

Solid.	Circumference.	Diameter.	$\frac{\text{Circumference}}{\text{Diameter}}$.
Wooden disc	12·2 cm.	3·9 cm.	3·13
Bottle	23·0 ,,	7·3 ,,	3·15
Gas jar	23·9 ,,	7·6 ,,	3·14
Iron ring	16·9 ,,	5·4 ,,	3·13
Table leg	34·5 ,,	10·9 ,,	3·16
Circular plate	39·9 ,,	12·7 ,,	3·14
Average value of Circumference÷Diameter=3·14			

43. Ratio of Circumference to Diameter. The results of Ex. 46 shew that the circumference of a circle contains its diameter approximately 3·14 times, and this is true of all circles. The determination of this ratio was the great problem of ancient mathematicians, and Archimedes (287-212 B.C.) shewed that its value lies between $3\frac{1}{7}$ and $3\frac{10}{71}$. We now know that the ratio cannot be found either as vulgar fraction or a terminating decimal, *i.e.* it is incommensurable. Its value to ten places is

$$3{\cdot}1415926535\ldots,$$

and it has actually been calculated to 707 places, but such a value is of no practical use. For most purposes it is sufficient to take either $3\frac{1}{7}$ or 3·14 as its value. It is usual to denote the ratio by the symbol π, which is the Greek form of the letter p, and is called *pi*. Thus for any circle,

$$\textbf{circumference} = \pi \times \textbf{diameter}.$$

If r be the radius, then diameter $=2r$, and

$$\textbf{circumference} = 2\pi r.$$

Ex. 47. *A circular racing track is* 10 *ch.* 4 *yd. in diameter; find the length of the track as a fraction of a mile, taking* $\pi = 3\frac{1}{7}$.

Here we have to find the circumference of a circle whose diameter is 10 ch. 4 yd., *i.e.* 224 yd.

Hence length of track $= \frac{22}{7} \times 224 = 22 \times 32$ yd.

$$= \frac{22 \times 32}{1760} = \frac{2}{5} \text{ or } 0{\cdot}4 \text{ of a mile.}$$

44. Length of an Arc. Any part of the circumference of a circle is called an **arc**, and the angle formed at the centre by the two radii drawn from the extremities of the arc is said to be the angle **subtended** at the centre by the arc. We shall now see how the length of an arc may be found.

PROBLEM 10. *To find the length of a circular arc.*

We know from geometry that in the same circle equal arcs subtend equal angles at the centre. It follows, therefore, that the lengths of the arcs of a circle are proportional to the angles which they subtend at the centre.

Now the whole circumference may be considered as an arc subtending an angle of 360° at the centre, so that if the given arc subtends an angle of $\theta°$, we have

360° is angle subtended by circumference;

$\therefore$ 1° ,, ,, ,, circumference $\div 360$,

and $\theta°$,, ,, ,, circumference $\times \theta \div 360$.

This is therefore the length of the arc subtending an angle of $\theta°$ at the centre,

i.e. **the length of an arc of a circle is measured by the fraction**

$$\frac{\textbf{angle of arc}}{\textbf{360}} \textbf{ of the whole circumference.}$$

If r be the radius of the circle, this becomes $\dfrac{\theta \,.\, 2\pi r}{360} = \dfrac{\theta\pi r}{180}$;

$$\therefore \textbf{ length of circular arc of radius r and angle } \theta = \frac{\theta\pi r}{180}.$$

Ex. 48. *Find the length of an arc subtending an angle of* 72° *at the centre of a circle of radius* 75 *cm.*

Here $\theta = 72°$, and $r = 75$ cm.;

$$\therefore \text{ length of arc} = \frac{72 \times 3{\cdot}14 \times 75}{180} = 94{\cdot}2 \text{ cm.}$$

45. Height of an Arc. Let PRQ (Fig. 46) be an arc of a circle whose centre is O. Draw the chord PQ and mark S, its mid-point. Join OP, OQ, and OS, producing the latter to meet the arc in R; then OS is perpendicular to PQ. The chord PQ is called the **chord of the arc,** whilst, if PR, RQ be joined, each of these is called a **chord of half the arc.** The distance SR is called the **height of the arc.** When the chord and the height of an arc are known, its radius may readily be found.

Ex. 49. *The height of a circular arc standing on a chord* 1 *ft.* 4 *in. long is* 5 *in. Find the radius of the arc.*

Referring to Fig. 46, SR = 5 in. and PQ = 16 in.;

$$\therefore \text{PS} = \tfrac{1}{2}\text{PQ} = 8 \text{ in.}$$

If r be the radius of the arc, OP = OR = OQ = r.

But $$\text{OP}^2 = \text{OS}^2 + \text{PS}^2 = (\text{OR} - \text{SR})^2 + \text{PS}^2;$$

$$\therefore r^2 = (r-5)^2 + 8^2 = r^2 - 10r + 89;$$

$$\therefore 10r = 89, \quad \text{or} \quad r = 8{\cdot}9 \text{ in.}$$

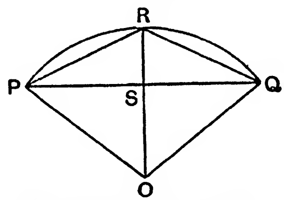


FIG. 46.—Height of an arc.

Ex. 50. *A circular arc standing on a chord* 10·5 *in. long has a radius of* 7·25 *in. Find the height of the arc.*

Referring again to Fig. 46,

$$\text{OP} = \text{OR} = 7{\cdot}25 \text{ in.} \quad \text{and} \quad \text{PS} = \tfrac{1}{2}\text{PQ} = 5{\cdot}25 \text{ in.}$$

Now $\text{OS}^2 = \text{OP}^2 - \text{PS}^2 = 7{\cdot}25^2 - 5{\cdot}25^2 = 12{\cdot}5 \times 2 = 25;$

$$\therefore \text{OS} = 5 \text{ in.}, \quad \text{and} \quad \text{SR} = \text{OR} - \text{OS} = 7{\cdot}25 - 5 = 2{\cdot}25 \text{ in.}$$

Ex. 51. *The chord of a circular arc of radius* 16·9 *cm. is* 24 *cm. long; find the length of the chord of half the arc.*

Referring once more to Fig. 46,

$$\text{OP} = \text{OR} = \text{OQ} = 16{\cdot}9 \text{ cm.}, \quad \text{and} \quad \text{PS} = \tfrac{1}{2}\text{PQ} = 12 \text{ cm.}$$

We require to find PR, but as only PS is known in the triangle PSR, we must first find the length of OS.

Now

$$OS^2 = OP^2 - PS^2 = 16{\cdot}9^2 - 12^2 = 28{\cdot}9 \times 4{\cdot}9 = \frac{289 \times 49}{100};$$

$$\therefore\ OS = \frac{17 \times 7}{10} = 11{\cdot}9 \text{ cm.};$$

$$\therefore\ SR = 16{\cdot}9 - 11{\cdot}9 = 5 \text{ cm.}$$

Hence $$PR^2 = PS^2 + SR^2 = 12^2 + 5^2 = 169;$$

$$\therefore\ PR = \mathbf{13} \text{ cm.}$$

46. General Relation. We can now easily find a general relation between the chord, height and radius of an arc.

PROBLEM 11. *To find the relation between the length of the chord of a circular arc, the height, and the radius of the arc.*

Let PQR (Fig. 46) be a circular arc of radius r, PQ its chord of length a, and SR its height h; then we have:

$$OP = OR = OQ = r, \quad SR = h, \quad \text{and} \quad PS = \tfrac{1}{2}PQ = \tfrac{1}{2}a.$$

Hence, since

$$OP^2 = PS^2 + OS^2 = PS^2 + (OR - SR)^2,$$

$$\therefore\ r^2 = \tfrac{1}{4}a^2 + (r-h)^2 = \tfrac{1}{4}a^2 + r^2 - 2hr + h^2,$$

so that $$2hr = \tfrac{1}{4}a^2 + h^2 = (a^2 + 4h^2)/4;$$

$$\therefore\ \mathbf{r = \frac{a^2 + 4h^2}{8h}}.$$

From this expression, it is evident that when any two of the quantities r, h, a are known, the third one may be found.

EXERCISES 7A.

Where no value of π is stated, it should be taken as $\frac{22}{7}$.

1. The diameter of a circular track is 1008 yards; find its circumference in miles.

2. How many laps to the mile are there on a circular racing track 140 yards in diameter?

3. How many kilometres are there in the circumference of a circle whose diameter is 364 yards, taking 8 kilometres as the equivalent of 5 miles?

4. Find the circumferences of the circles whose radii are 8·75 in., 10·5 in., and 15·75 in. respectively. Calculate the diameter of the circle whose circumference is equal to the sum of the circumferences of these three circles.

5. A circle of wire has a diameter of 9·8 inches, and the wire is bent into the form of a square; find the length of a side of the square.

6. The radius of a circle measured in inches is 5·25; the circumference measured in centimetres is 83·82. Find the number of centimetres equivalent to an inch.

7. The circumference of a circle is 155·87 yards and its radius is 2275 centimetres; find the number of yards equivalent to a metre.

8. Three sides of a plot of ground form sides of a square each 17·5 yards long; the fourth side is exactly a semicircle. Find the length of fencing required to enclose the plot completely.

9. A semicircle is described on one side of a line and a rectangle of breadth 22·5 inches on the other side. The perimeter of the whole figure thus formed is 4 yards. Find the radius of the semicircle.

10. Two concentric circles are such that the circumference of the smaller is 0·64 that of the larger. Find the radius of the larger if that of the smaller is 1 ft. 4 in.

11. A small circle whose diameter is 13 inches rolls round the inside of a larger circle of 2 ft. radius once in $2\frac{1}{2}$ seconds. Find the velocity of its centre in miles per hour.

12. The driving wheel of an express engine is 6 ft. 5 in. in diameter. How many times will this wheel revolve in travelling 16·5 miles?

13. A bicycle wheel revolves 90 times in a furlong and 447 times in a kilometre. Deduce the number of centimetres in an inch to two decimal places. (L.S.)

14. The driving wheels of a locomotive are 7 ft. in diameter, and the other wheels are 3 ft. 9 in. in diameter. Find in miles the length of a journey during which the smaller wheels make 13,000 more revolutions than the driving wheels. (C.S)

15. A cyclist travelling at 12 miles per hour round a circular track goes completely round 8 times in 4 minutes. Find the diameter of the track in yards.

16. A large panel with equal semicircular ends has a perimeter of 24 ft. The length of the straight edges is just three times its width; find the radius of each semicircle and the length of the straight edges.

17. On the four sides of a rectangle semicircles are drawn outside the rectangle. The perimeter of the figure thus formed is 8 ft. 3 in. The sides of the rectangle are as 5 to 4; find the radii of the semicircles.

18. The indicator on a locomotive shews that the driving-wheel, whose diameter is 8 ft. 4 in., is making 147 revolutions per minute. Find the velocity of the engine in miles per hour.

19. What must be the shortest diameter of a circular shaft in order that two rectangular cages may pass in it, each cage being 11 ft. long by 8 ft. 8 in. wide? Allow 16 inches between the cages for clearance and 14 inches between the corners of the cages and the walls of the shaft.

20. The circumference of the hind wheel of a wagon exceeds that of the front wheel by 3 ft. 8 in., and in travelling a mile the front wheel makes 120 revolutions more than the hind wheel. Find the diameters of the wheels.

21. ABCD is a square and AEC a circular arc whose centre is D and radius DA. The figure enclosed between AB, BC and the arc AEC is called a fillet. Find the radius of the circular edge of a fillet whose perimeter is to be 37·5 in.

22. The indicator of a locomotive shews that the driving wheel, whose diameter is 9 ft. 4 in., is making 135 revolutions per minute. Find the velocity of the engine in miles per hour. If one of the smaller wheels makes 6 revolutions per second, find its diameter.

In Exercises 23-31, *contracted methods are to be used, taking* $\pi = 3{\cdot}1416$, *and the results given correctly to two places of decimals unless otherwise stated.*

23. Find the circumference of a circle whose radius is 8·62 cm.

24. Calculate the distance a wheel will roll in making 37·23 revolutions, if its diameter is 2·75 ft.

25. Find the radius of a circle whose circumference is 89·73 cm.

26. A bicycle wheel is 28 inches in diameter. How many revolutions to the nearest unit will it make in travelling 1 kilometre? Take one inch = 2·5400 cm. (L.M.)

27. The front and back wheels of a bicycle have diameters 30 in. and 28 in. respectively. Find to the nearest yard how far the bicycle must be ridden in order that the back wheel may make 200 more revolutions than the front wheel. (C.S.)

28. Find the length of wire in a circular coil in which there are 83·3 turns of average diameter 5·579 ft.

29. 4009 cm. of wire are wound on a bobbin of average diameter 15·3 cm. Regarding each turn of wire as circular, find, to one decimal place, the number of turns on the bobbin.

30. Find the diameter in yards of a circular track on which there are 3·6 laps to the mile.

31. A wire bent in the form of a rectangle 9·23 ft. long by 7·86 ft. broad is made into a circle. Find the radius of this circle.

32. A saw band is to run over two equal pulleys like a belt. The pulleys are each 5 ft. in diameter, and their centres are 5·4 ft. apart. Find the shortest length of saw blade required. Take $\pi = 3{\cdot}14$.

33. Find the length of an arc of a circle of radius 2·7 ft. subtending an angle of 28° at the centre.

34. An arc subtends an angle of two-thirds of a right angle at the centre of a circle whose diameter is 4·2 m. Find its length.

35. Find the length of an arc which subtends an angle of 135° at the centre of a circle of diameter 11·2 yards.

36. What angle must be subtended at the centre of a circular curve 9·72 chains in radius by a rail 59·4 links long?

37. What angle is subtended by an arc of a circle equal in length to its radius?

38. Find the angle subtended by an arc of a circle which is 1·1 times the length of its radius.

39. Find the radius of a circle in which an arc 14·3 ft. long subtends an angle of 91° at the centre.

40. A circular track is 528 metres in diameter and part of its boundary 29·2 chains in length subtends an angle of 128° at the centre. Find the number of yards in a metre.

41. P is the mid-point of the arc APB of a circle whose centre is O; the chord AB cuts OP at right angles in Q. If AB = 11·2 cm., PQ = 3·2 cm., find the radius OP.

42. An arch in the form of a segment of a circle is to have a span of 36 inches and a height of 12 inches. Calculate the radius of the arch. (U.L.C.I.)

43. Find the height of an arc standing on a chord 33·6 cm. long, the radius of the arc being 19·3 cm.

44. The chord of half an arc is 10·1 cm. in length and the height of the arc is 2 cm.; find the length of the chord and the radius of the arc.

45. ABC is a circle whose centre is O, OA is a radius which is produced to a point H, and HB is the tangent drawn from H. If the circle represents the earth, whose diameter is 7920 miles, and HB = 89 miles, find AH in yards.

46. AB is a chord of a circle whose centre is O, and C is a point in AB such that AC = 36 cm., CB = 64 cm. The radius of the circle is 73 cm., find the length of OC.

47. In a circle, a chord of length l cuts off a segment of altitude h; find an expression for the diameter d of the circle in terms of h and l. (U.L.C.I.)

Find h when $d = 35$ and $l = 21$.

48. The radius of a circle is 16·9 in., and the height of an arc cut off by a chord is 5 in. Find the length of the chord and also that of the chord of half the arc.

49. In a circle the chord of half the arc is 1 cm. longer than 4 times the height of the arc, and 2 cm. longer than half the chord upon which the arc stands. Find the lengths of the chord, the height and the radius of the arc.

50. Two pulleys of diameters 8·6 ft. and 4 ft. respectively, whose centres are 12·5 ft. apart, are to be connected by an endless belt; calculate the minimum length of belting required.

47. Area of a Circle. Since a circle is bounded by a curved line, we may find its approximate area either by the Trapezoidal Rule (p. 51) or by Simpson's Rule (p. 52).

Ex. 53. *Find* (i) *by the Trapezoidal Rule and* (ii) *by Simpson's Rule, the approximate area of a circle whose radius is* 4·8 *cm. Also compare the area with the area of the square on the radius.*

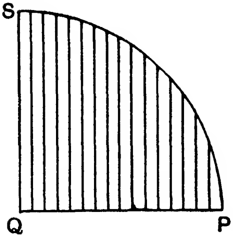


FIG. 47.—Area of a circle.

Take a sheet of paper accurately ruled in millimetre squares, and on it describe a circle of radius 4·8 cm. Draw two perpendicular diameters; then the circle is divided into four equal areas called **quadrants**. It will only be necessary to determine the area of one of these quadrants.

Let PQS (Fig. 47) be one quadrant. Divide PQ into 16 equal parts, and draw the ordinates at the points of division. Read off the lengths of these ordinates, and record the results as follows:

To Q.	End Ordinates.	Even Ordinates.	Odd Ordinates.
48 mm.	48 mm.		
45 „		47·9 mm.	
42 „			48·6 mm.
39 „		47·2 „	
36 „			46·5 „
33 „		45·6 „	
30 „			44·5 „
27 „		43·2 „	
24 „			41·6 „
21 „		39·7 „	
18 „			37·5 „
15 „		34·9 „	
12 „			31·7 „
9 „		28·0 „	
6 „			23·2 „
3 „		16·7 „	
0 „	0		
From P			
	48 mm.	303·2 mm.	273·6 mm.

Hence (i) by the Trapezoidal Rule,

$$\text{area of quadrant} = 3(24 + 303{\cdot}2 + 273{\cdot}6) = 1802{\cdot}4 \text{ sq. mm.};$$

$\therefore$ area of whole circle $= 1802{\cdot}4 \times 4 = 7209{\cdot}6$ sq. mm.

$= \mathbf{72{\cdot}1}$ sq. cm. nearly.

(ii) By Simpson's Rule,

$$\text{area of quadrant} = \tfrac{1}{3} \,.\, 3(48 + 303{\cdot}2 \times 4 + 273{\cdot}6 \times 2)$$
$$= 1808 \text{ sq. mm.};$$

$\therefore$ area of whole circle $= 1808 \times 4 = 7232$ sq. mm. $= \mathbf{72{\cdot}3}$ sq. cm.

Now, area of square on radius $= 4{\cdot}8 \times 4{\cdot}8 = 23{\cdot}04$ sq. cm.

And taking 72·32 sq. cm. as the more accurate value of the area of the circle,

$$\frac{\text{area of circle}}{\text{area of square on radius}} = \frac{72{\cdot}32}{23{\cdot}04} = \mathbf{3{\cdot}14}.$$

EXERCISES 7B.

The following measurements give the diameters of twelve circles. In each case calculate the area by the Trapezoidal and Simpson's Rules, and compare the area with that of the square on the radius.

1. 46·5 cm.	**2.** 49·25 in.	**3.** 64 cm.	**4.** 67·25 cm.
5. 39·25 in.	**6.** 33 in.	**7.** 6·25 ft.	**8** 45·5 ft.
9. 80·25 cm.	**10.** 2 yd.	**11.** 83·5 cm.	**12.** 97·25 cm.

48. Ratio of Areas of a Circle and the Square on its Radius. The results of the above exercises shew that the ratio of the area of a circle to that of the square on its radius comes to the value of π approximately in all cases. We shall now see whether this is true generally.

PROBLEM 12. *To find the area of a circle.*

Let ABCD ... (Fig. 48) be a regular polygon of n sides inscribed in a circle whose centre is O. Join each vertex to O and draw perpendiculars from O to each side of the polygon. These meet the sides at their points of contact with the circle, and are thus equal in length to the radius of the circle.

The polygon is divided in n equal triangles OAB, OBC, ... , all of which have equal bases AB, BC, ... , and equal heights.

Let r = radius of circle, and s = length of one of the equal sides of polygon; then

$$\begin{aligned}\text{area of polygon} &= n\,.\,(\text{area of one triangle OAB})\\ &= n\,.\,\tfrac{1}{2}rs = \tfrac{1}{2}r\,.\,ns\\ &= \tfrac{1}{2}r(\text{perimeter of polygon}).\end{aligned}$$

This is true, however large n may be, and by increasing n, we can make the perimeter of the polygon differ by as little as we please from the circumference of the circle. Hence when n is increased indefinitely, the area of the polygon becomes the area of the circle, so that

$$\text{area of circle} = \tfrac{1}{2}r(\text{circumference}) = \tfrac{1}{2}r(2\pi r) = \pi r^2.$$

$\therefore$ **the area of a circle is measured by the product, π(radius)^2.**

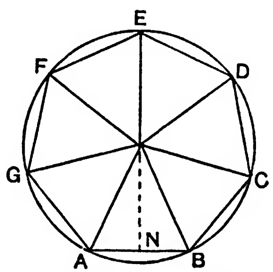


FIG. 48.—Area of a circle.

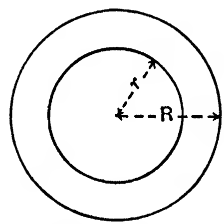


FIG. 49.—Area of a ring.

Ex. 53. *Find the acreage of a circular plot of land whose diameter is* 10 *chains* 50 *links.*

Radius of plot = 5·25 ch.;

$\therefore$ area = π . 5·25 × 5·25 sq. ch. = 8·66 acres.

49. Area of an Annulus or Circular Ring. An annulus or circular ring is the figure bounded by two concentric circles (see Fig. 49). The area of such a figure is clearly the difference between the areas of the two circles.

Ex. 54. *A path* 3 *yd.* 2 *ft. wide surrounds a circular lake whose diameter is* 38 *yd.* 1 *ft. Find the area of the path as a fraction of an acre.*

Radius of outer circular boundary = radius of lake + width of path = $(38\frac{1}{3} \div 2) + 3\frac{2}{3} = 22\frac{5}{6}$ yd.;

$$\therefore \text{ Area of path and lake} = \pi \times 22\tfrac{5}{6} \times 22\tfrac{5}{6} \text{ sq. yd.,}$$
$$\text{and area of lake} = \pi \times 19\tfrac{1}{6} \times 19\tfrac{1}{6} \text{ sq. yd.;}$$
$$\begin{aligned}\therefore \text{ Area of path alone} &= \pi(22\tfrac{5}{6})^2 - \pi(19\tfrac{1}{6})^2\\ &= \pi\{(22\tfrac{5}{6})^2 - (19\tfrac{1}{6})^2\}\\ &= \pi(22\tfrac{5}{6} + 19\tfrac{1}{6})(22\tfrac{5}{6} - 19\tfrac{1}{6})\\ &= \pi \,.\, 42 \,.\, 3\tfrac{2}{3} \text{ sq. yd.}\\ &= \frac{22 \times 42 \times 11}{7 \times 4840 \times 3} \text{ acre} = \mathbf{0{\cdot}1} \text{ acre.}\end{aligned}$$

Problem 13. *To find an expression for the area of an annulus bounded by two concentric circles of radii* R *and* r *respectively.*

Let $R > r$, then

$$\text{area of larger circle} = \pi R^2.$$

and
$$\text{area of smaller circle} = \pi r^2,$$
$$\therefore \text{ area of annulus} = \pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi(R + r)(R - r),$$

i.e. **area of an annulus bounded by two concentric circles**

$$= \pi \times (\textbf{sum of radii}) \times (\textbf{difference of radii}).$$

The above result may be written also in the form

$$2\pi\left(\frac{R+r}{2}\right)(R - r).$$

Now $\frac{1}{2}(R + r)$ is the average radius, and $R - r$ is the breadth; hence, **area of annulus $= 2\pi \times$ (average radius) $\times$ (breadth).**

Ex. 55. *The outer boundary of a path* 10·5 *ft. wide surrounding a circular lake is one-seventh of a mile long. Find the diameter of the lake and the area of the path.*

Let the radius of the lake be r ft., then the radius of the outer boundary of path is $(r + 10{\cdot}5)$ ft.,

$\therefore$ circumference of outer boundary $= 2\pi(r + 10{\cdot}5)$ ft., and this is $\frac{1}{7}$ mile $= 1760 \times 3/7$ ft.;

$$\therefore\ 2\pi(r + 10{\cdot}5) = (1760 \times 3)/7 \text{ ft.;}$$
$$\therefore\ r + 10{\cdot}5 = \frac{1760 \times 3 \times 7}{7 \times 2 \times 22} = 120 \text{ ft., taking } \pi = \tfrac{22}{7};$$
$$\therefore\ r = 120 - 10{\cdot}5 = 109{\cdot}5 \text{ ft.;}$$

hence, diameter of lake $= 2r = 219$ ft. or **73** yd.

$$\begin{aligned}\text{Also, area of path} &= \pi(120^2 - 109{\cdot}5^2) \text{ sq. ft.}\\ &= \tfrac{22}{7} \,.\, 229{\cdot}5 \,.\, 10{\cdot}5\\ &= \mathbf{841{\cdot}5} \text{ sq. yd.}\end{aligned}$$

EXERCISES 7c.

Find the area of each of the circles whose radii are given below; take $\pi=\frac{22}{7}$.

1. 10·5 inches.	2. 2·1 feet.	3. 17·5 cm.
4. 9·1 yards.	5. 3·5 chains.	6. 12·6 metres.

Find the area of each of the circles whose diameters are given below, taking $\pi=3{\cdot}1416$, and giving the results correct to two places of decimals.

7. 3·5 in.	8. 5·25 cm.	9. 6·75 ft.
10. 12·25 in.	11. 34 cm.	12. 2·7 chains.

Find the diameter of each of the circles whose areas are given below, taking $\pi=\frac{22}{7}$.

13. 38·5 sq. ft.	14. 346·5 sq. cm.	15. 3·85 ac.	16. 0·55 ac.

Find the diameter of each of the circles whose areas are given below, taking $\pi=3{\cdot}14$.

17. 2122·64 sq. cm.	18. 1352·7 sq. in.	19. 975·91 sq. ft.
20. 2874·8 sq. yd.	21. 4329·9 sq. cm.	22. 7539·14 sq. m.

23. The area of a circle is 10 sq. in. Find the circumference to the nearest tenth of an inch. Take $\pi=3{\cdot}14$. (C.S.)

24. Five times round a circular race-course measures one mile. Find, to the nearest square yard, the area of the space enclosed. Take $\pi=3{\cdot}1416$. (C.S.)

25. A circular flower-bed has an area of 158 square feet. Find its diameter to the nearest inch. Take $\pi=3{\cdot}1416$. (J.M.B.)

Where no value of π is stated, it should be taken as $\frac{22}{7}$.

26. ABC is a triangular garden whose side AB is 6·6 metres long, and the perpendicular distance from C to AB is 4·2 metres long. Find the diameter of a circle whose area is equal to that of the triangle.

27. Out of a large circular disc whose diameter is 7·8 inches, another is cut whose area is $\frac{4}{9}$ths that of the disc. Find the radius of the one cut out, taking $\pi=3{\cdot}14$.

28. A circular plot of ground contains 337·55 acres, and its diameter is 1·32 kilometres; calculate the number of square feet in a square metre to two places of decimals.

29. If a circle be inscribed in a square of side 3 inches, so as to touch each side at its middle point, find (i) the area of one of the corner-pieces between the circle and the square, and (ii) the length of the curved side of this corner-piece. (J.M.B.)

30. The shape of a flat metal plate is an equilateral triangle surrounded by three semicircles, each described on one of the sides as diameter. The area of the plate is 58 square inches. Find the approximate length of a side of the triangle, being given that the area of an equilateral triangle is 0·433 times the square of a side and the area of a circle $= \frac{11}{14}$ of the square on its diameter. (L.S.)

31. Find the cost of paving a gravel walk ten feet wide surrounding a circular pond whose diameter is 492 ft., at the rate of 4s. 6d. per square yard. Take $\pi = 3{\cdot}14$.

32. The cost of paving a uniform path surrounding a circular pond 60 feet in diameter is £193 17s. 6d. Find the width of the path, taking the average cost per square yard at fifteen shillings.

33. Find, correct to four significant figures, the area between two concentric circles whose radii are 2·3 and 3·4 inches respectively. Take $\pi = 3{\cdot}1416$. (C.S.)

34. Find the width of a path surrounding a circular garden 220 ft. in circumference, when the area of the path is 0·44 that of the garden.

35. Shew that the area A of an annulus of width w, and greatest diameter d is given by

$$A = \pi w(d - w).$$

Hence find w when $d = 84$ and $A = 1694$.

36. On a line 4 ft. 8 in. long a quadrant and a semicircle are described on the same side. Find the area bounded by these two curves.

37. ABC is a triangle right-angled at C. Semicircles ADCEB, CFB, AGC are described on AB, BC, CA respectively, the points F, G being outside the triangle. If BC $= 4{\cdot}8$ cm., CA $= 5{\cdot}5$ cm., find the sum of the curvilinear areas CFBE, AGCD. What may be inferred from the result?

38. The diameter PQ of a circle is divided into three equal parts, PL, LM, MQ. On PL, PM semicircles are described on the upper side of the lines, and on QM, QL semicircles are described on the lower side of the lines, thus forming a curvilinear area PMQL. Prove that this area is equal to one-third of that of the circle.

39. OPQ is an equilateral triangle, OR the perpendicular from O to PQ. PQ is produced to S so that PS $= 3 \cdot$ OR, and RO is produced to T so that OT $=$ RO. If RO be one unit in length, calculate the length of ST, and shew that ST^2 is approximately equal to π^2, taking $\pi = 3{\cdot}1416$.

CHAPTER VIII

CIRCULAR MEASURE. AREAS OF CIRCULAR SECTORS AND SEGMENTS

50. Ratio of Length of Arc to its Radius. In many problems on areas it is necessary to use another method of measuring angles. This will now be investigated.

Ex. 56. *Find the ratio of the length of an arc to that of its radius.*

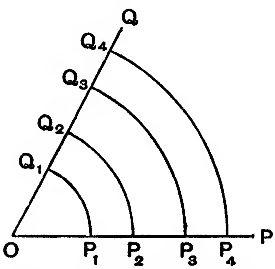


FIG. 50.—Ratio of arc to radius.

Draw any angle POQ (Fig. 50), and on OP take a number of points P_1, P_2, P_3, ... With centre O and radii OP_1, OP_2, OP_3 ... respectively, describe arcs cutting OQ in Q_1, Q_2, Q_3, ... Measure the length of each arc by stepping a piece of cotton carefully round it. Measure also the length of the corresponding radius; then divide each length of arc by that of its radius. Tabulate the results as follows:

Length of Arc.	Length of Radius.	Arc ÷ Radius.
$P_1Q_1 = 5{\cdot}5$ cm.	$OP_1 = 5$ cm.	1·10
$P_2Q_2 = 8{\cdot}7$,,	$OP_2 = 8$,,	1·09
$P_3Q_3 = 12{\cdot}5$,,	$OP_3 = 11{\cdot}5$,,	1·09
$P_4Q_4 = 15{\cdot}3$,,	$OP_4 = 14$,,	1·09
$P_5Q_5 = 19{\cdot}6$,,	$OP_5 = 18$,,	1·09
Average ratio of Arc to Radius =		1·09
Measurement of Angle POQ = 62·5°.		

Repeat the exercise with different angles.

51. Circular Measure. The above results shew that for each angle the ratio arc/radius is constant. Indeed, this fact may be directly deduced from the expression for the length of an arc, already found in Prob. 10 (p. 80). There it was shewn that the length of a circular arc of radius r and angle θ is $\frac{\pi r \theta}{180}$.

Hence, $$\frac{\text{length of arc}}{r} = \frac{\pi\theta}{180}.$$

Thus, putting $\theta = 62{\cdot}5$,

$$\frac{\text{length of arc}}{r} = \frac{3{\cdot}14 \times 62{\cdot}5}{180} = 1{\cdot}09,$$

which agrees with the value found in Ex. 56.

In a similar way, if $\theta = 51°$, it will be found that arc/radius $= 0{\cdot}9$; if $\theta = 23°$, the ratio $= 0{\cdot}4$, and so on, each angle having its own ratio. This ratio may therefore be taken as a measure of the angle, and such a system of measurement is known as **Circular Measure.** For the ratio to be unity, it is clear that the length of arc must be equal to that of its radius. **The angle subtended at the centre of a circle by an arc equal in length to its radius** is taken as the unit angle, and is called a **radian.**

Hence $62{\cdot}5° = 1{\cdot}09$ radians, $51° = 0{\cdot}9$ radian, $23° = 0{\cdot}4$ radian, and so on.

Ex. 57. *What is a radian? Find the number of radians in a right angle; hence determine the size of a radian in degrees and use this value to find the number of degrees in* $2{\cdot}0944$ *radians. Take* $\pi = 3{\cdot}1416$.

A radian, as just defined, is the angle subtended at the centre of a circle by an arc equal in length to that of its radius. It should be carefully observed that the radius of any circle may be marked off on the circumference exactly six times *with a pair of compasses*, but the distance between any two consecutive points thus marked is measured *in a straight line, i.e.* along the chord joining those points. The chord and its arc, therefore, subtend an angle of $360° \div 6 = 60°$ at the centre. The radian, however, is subtended by an arc *equal in length to its radius*, this length being measured *along the arc.* The angle is therefore less than 60°.

Let r be the radius of any circle, then the length of arc subtending a right angle at the centre $=2\pi r/4=\frac{1}{2}\pi r$.

$$\therefore \text{ Number of radians in } 90^\circ=\frac{\pi r}{2r}=\frac{\pi}{2}=1\cdot5708.$$

Hence $\qquad 1 \text{ radian}=90\div1\cdot5708=57\cdot3^\circ$ nearly.

And $\qquad 2\cdot0944 \text{ radians}=2\cdot0944\times57\cdot3=120^\circ$.

52. Important Relations and Values. It is clear that the number of radians in an angle θ° subtended at the centre of a circle of radius r is equal to

$$\frac{\text{length of arc}}{r}=\frac{\pi\theta}{180};$$

$$\therefore\ \theta \textbf{ degrees are equivalent to } \frac{\pi\theta}{180} \textbf{ radians.}$$

Putting $\theta=90^\circ$, 180°, and 1° respectively, we get

$$90^\circ=\frac{\pi}{2}\text{ radians,}\quad 180^\circ=\pi\text{ radians,}\quad 1^\circ=\frac{\pi}{180}=0\cdot01745\text{ radian.}$$

From the last relation, we have also

$$1\text{ radian}=\frac{180}{\pi}=57\cdot2958^\circ=57^\circ\ 14'\ 45''\ \text{ or }\ 57\cdot3^\circ\text{ nearly.}$$

The last value will generally be sufficient.

In working with radians, the following values are useful:

$$\pi=3\cdot1416,\quad \frac{1}{\pi}=0\cdot3183.$$

53. Circular Sector. The figure bounded by two radii of a circle and the arc between them is called a **Sector** of the circle; thus POQ (Fig. 51) is a circular sector. The angle POQ subtended at the centre by the arc is called the **angle of the sector.**

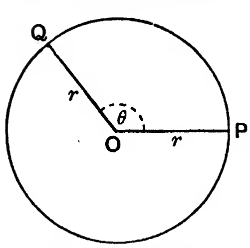


FIG. 51.—Circular sector.

PROBLEM 14. *Find an expression for the area of a circular sector in terms of its radius and its angle.*

Let POQ (Fig. 51) be a circular sector whose radius is r and angle POQ $=\theta$ radians; then it is evident that the area of the sector is a fraction of that of the circle.

Let the whole circumference be divided into n equal parts, where n is an integer sufficiently large that an exact number of these parts is contained in the arc PQ. By joining the points of division to the centre O, the circle will be divided into n sectors of equal area, each having an angle of $2\pi/n$ radians.

Hence the number of such sectors in the sector POQ is $\theta \div 2\pi/n$, *i.e.* $\frac{\theta}{2\pi} \cdot n$, so that, if a be the area of one of the n sectors, then area of sector POQ $= \frac{\theta}{2\pi} \cdot na$.

But the sum of the n areas = area of whole circle $= \pi r^2$;

$$\therefore \; na = \pi r^2,$$

and $$\text{area of sector POQ} = \frac{\theta}{2\pi} \cdot na = \frac{\theta}{2\pi} \cdot \pi r^2 = \tfrac{1}{2}\theta r^2,$$

i.e. **the area of a circular sector is measured by the product**

$$\tfrac{1}{2}(\text{radius})^2 \times (\text{angle of sector in radians}).$$

If s be the length of the arc PQ, then $\theta = s/r$, and

$$\textbf{area of sector} = \tfrac{1}{2}r^2 s/r = \tfrac{1}{2}rs.$$

Ex. 58. *The angle of a sector is 27° 30′, and its radius is 7 ft. 6 in.; find its perimeter and its area.*

The perimeter = twice radius + length of arc

$$= 15 + \frac{\pi \times 7{\cdot}5 \times 27{\cdot}5}{180} = 15 + 3{\cdot}6 = \mathbf{18{\cdot}6} \text{ ft.}$$

Now the number of radians in $27° \, 30' = \frac{27{\cdot}5 \times \pi}{180} = 0{\cdot}48$;

$$\therefore \; \text{area of sector} = \tfrac{1}{2} \,.\, 7{\cdot}5^2 \times 0{\cdot}48 = \mathbf{13{\cdot}5} \text{ sq. ft.}$$

Since the length of the arc = 3·6 ft. as found above, we might have used it to find the area, thus

$$\text{area of sector} = \tfrac{1}{2} \,.\, 7{\cdot}5 \,.\, 3{\cdot}6 = 13{\cdot}5 \text{ sq. ft. as before.}$$

Ex. 59. *The perimeter of a sector is equal to that of the square described on one of its bounding radii. Find the angle of the sector in degrees, and shew that the area of the sector is equal to that of the square.*

Let r be the radius and θ the angle in radians of the sector, then the length of its arc $= r\theta$;

$$\therefore \; \text{its perimeter} = 2r + r\theta.$$

But the perimeter of the square described on its radius $=4r$;

$$\therefore\ 2r+r\theta=4r,$$

or
$$\theta=2 \text{ radians}=\frac{360}{\pi} \text{ degrees}$$
$$=114{\cdot}6^\circ$$

Now, area of sector $=\frac{1}{2}r^2\theta=r^2=$ area of square on radius, since $\theta=2$.

54. Segment of a Circle. The figure bounded by the chord of a circle and the arc it cuts off is called a **segment** of a circle. Thus PAQ, PBQ (Fig. 52) are the two segments into which the chord PQ divides the circle. The arc PAQ, which is the smaller, is called the **minor arc**, whilst the larger arc PBQ is called the **major arc**. The segments PAQ, PBQ are similarly distinguished as the minor and major segments respectively.

PROBLEM 15. *Find an expression for the area of a circular segment.*

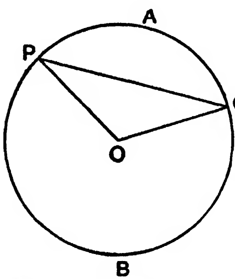


FIG. 52.—Segment of a circle.

Let PAQ (Fig. 52) be a circular segment, and suppose its arc PAQ subtends an angle of θ radians at the centre O, and that its radius is r; then the sector PAQO is made up of the triangle POQ and the segment PAQ.

Now the area of the sector is, by Prob. 14, $\frac{1}{2}r^2\theta$, and the area of the triangle POQ $=\frac{1}{2}$. OP . OQ . sin POQ $=\frac{1}{2}r^2 \sin\theta$.

Hence, if a be the area of the segment PAQ,
$$\tfrac{1}{2}r^2\theta=a+\tfrac{1}{2}r^2\sin\theta;$$
$$\therefore\ a=\tfrac{1}{2}r^2(\theta-\sin\theta),$$

i.e. **the area of a segment whose arc subtends an angle of θ radians at its centre and whose radius is r $=\frac{1}{2}$r$^2(\theta-\sin\theta)$.**

It should be observed that the area of the major segment is
$$\pi r^2-\tfrac{1}{2}r^2(\theta-\sin\theta)=\tfrac{1}{2}r^2(2\pi-\theta+\sin\theta).$$

Ex. 60. *The radius of the arc of a circular segment is* 2 *ft.* 1 *in., and the angle subtended by it at the centre is* 34° 24′. *Find the area of the segment and the length of its chord.*

From the tables we have sin 34° 24′ $=0{\cdot}5650$, and

$$34^\circ\,24'=\frac{\pi\times 34{\cdot}2}{180}=0{\cdot}6004 \text{ radian};$$

$\therefore$ area of segment $= \frac{1}{2} \cdot 25^2(0{\cdot}6004 - 0{\cdot}5650)$ sq. in.
$= \frac{1}{2} \cdot 625 \cdot 0{\cdot}0354 = \mathbf{11{\cdot}06}$ sq. in.

Let l inches = length of chord, then

$$\frac{l}{2r} = \sin \text{(half the angle of segment)}$$
$$= \sin 17° \, 12' = 0{\cdot}2957.$$

But $r = 25$ inches;

$\therefore$ $l = 0{\cdot}2957 \times 50$ inches $= 14{\cdot}785$ in. $= \mathbf{14{\cdot}8}$ in.

55. Approximate Length of an Arc. We have already seen that the length of a circular arc of radius r, subtending an angle of θ radians at the centre is $r\theta$. In many practical problems, however, when θ is not large and is unknown, the length of a circular arc is calculated by the following approximate rule, first given by Christian Huyghens, a Dutch scientist, about 1651.

Let l = length of arc, a = length of its chord, and b = length of the chord of half the arc, then

$$\mathbf{l = (8b - a)/3.}$$

Ex. 61. *The chord of an arc is* 36 *cm. long, and the height of the arc is* 1·9 *cm.; find*

(*a*) *the length of the chord of half the arc,*
(*b*) *the radius of the arc, and*
(*c*) *the angle subtended at the centre.*

Hence find the true length and the approximate length of the arc, using Huyghens' rule for the latter.

Referring to Fig. 46, p. 81,

(*a*) $\mathrm{PS} = \frac{1}{2}\mathrm{PQ} = 18$ cm., $\mathrm{SR} = 1{\cdot}9$ cm.,

and $\mathrm{PR}^2 = \mathrm{PS}^2 + \mathrm{SR}^2 = 18^2 + 1{\cdot}9^2 = 327{\cdot}61$;

$\therefore$ $\mathrm{PR} = \sqrt{327{\cdot}61} = \mathbf{18{\cdot}1}$ cm.

(*b*) Let radius of arc $= r$ cm., then since $\mathrm{OP}^2 = \mathrm{PS}^2 + \mathrm{SO}^2$,

$\therefore$ $r^2 = 18^2 + (r - 1{\cdot}9)^2$,

from which $3{\cdot}8r = 327{\cdot}61$, or $r = \mathbf{86{\cdot}21}$ cm.

(*c*) Let $\angle \mathrm{POQ} = \theta$, then $\angle \mathrm{POR} = \frac{1}{2}\theta$,

and $\sin \frac{1}{2}\theta = \mathrm{PS}/\mathrm{OP} = 18/86{\cdot}21 = 0{\cdot}2088$;

$\therefore$ from the tables, $\frac{1}{2}\theta = 12° \, 3'$

and $\theta = \mathbf{24° \, 6'} = \mathbf{0{\cdot}4206}$ radian.

Hence, length of arc $= r\theta = 86{\cdot}21 \times 0{\cdot}4206 = \mathbf{36{\cdot}26}$ cm.

But by Huyghens' Rule,

$$a = \mathsf{PQ} = 36 \text{ cm.}, \quad b = \mathsf{PR} = 18{\cdot}1 \text{ cm.};$$

$$\therefore \text{ length of arc} = (18{\cdot}1 \times 8 - 36)/3 = \mathbf{36{\cdot}27} \text{ cm.}$$

The two results thus differ by 0·01 cm.

56. Approximate Area of a Segment. Several expressions giving the approximate area of a circular segment are in practical use. These involve measurements which can be made most conveniently. The chief of these approximations are:

(i) When the height h and the radius r of the arc of the segment are known, its approximate area is

$$\frac{4h^2}{3}\sqrt{\frac{2r}{h} - 0{\cdot}608}.$$

(ii) When the height h and the length a of the bounding chord of the segment are known, its approximate area is

$$\frac{h(4a^2 + 3h^2)}{6a}.$$

Ex. 62. *The chord of a circular segment is* 14 *inches long, and the height of the arc is one inch. Find* (*a*) *the radius of the arc, and* (*b*) *the angle it subtends at the centre.*

Hence calculate the area of the segment (i) *accurately to three places of decimals, and* (ii) *approximately by the above rules.*

Again, referring to Fig. 46, p. 81,

(*a*) $\mathsf{PS} = \frac{1}{2} . \mathsf{PQ} = 7$ in., and $\mathsf{SR} = 1$ in.

Let radius of arc be r inches, then since $\mathsf{OP}^2 = \mathsf{PS}^2 + \mathsf{SO}^2$, $r^2 = 49 + (r-1)^2$, from which $r = \mathbf{25}$ in.

(*b*) Let $\theta = \angle \mathsf{POQ}$, then

$$\sin \tfrac{1}{2}\theta = \sin \mathsf{POS} = \mathsf{PS/OP} = 7/25 = 0{\cdot}28;$$

$$\therefore \text{ from the tables,} \qquad \tfrac{1}{2}\theta = 16^\circ\, 16';$$

$$\therefore \theta = \mathbf{32^\circ\, 32'} = \mathbf{0{\cdot}5678} \text{ radian.}$$

From Problem 15,

$$\text{area of segment} = \tfrac{1}{2}r^2(\theta - \sin\theta) = \tfrac{1}{2} . 625(0{\cdot}5678 - 0{\cdot}5378)$$

$$= \tfrac{1}{2} . 625 \times 0{\cdot}03 = \mathbf{9{\cdot}375} \text{ sq. in.}$$

By the approximate rule (i), $h=1$, $r=25$;

$\therefore$ area of segment $=4\sqrt{49 \cdot 39}/3 = \mathbf{9 \cdot 371}$ sq. in.

By the approximate rule (ii), $a=14$, $h=1$;

$\therefore$ area of segment $=(4 \times 14^2+3)/84 = 787/84 = \mathbf{9 \cdot 369}$ sq. in.

57. An Important Application. The facts we have already learned about circular arcs are of great importance in setting out a railway curve.

Let $T_0T_1T_2T_3$ (Fig. 53) be the plan of a section of a railway curve which has to be constructed, O being the centre. The arcs

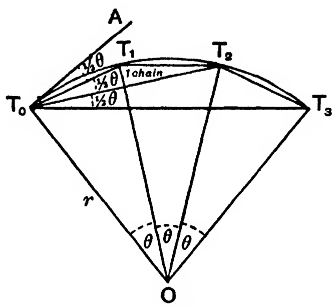


FIG. 53.—Railway curves.

T_0T_1, T_1T_2, T_2T_3 are equal, so that $\angle T_0OT_1 = \angle T_1OT_2 = \angle T_2OT_3$. Denote each of the angles by θ, and suppose the radius of the arc is r.

Draw T_0A, the tangent to the arc at T_0, then

$$\angle AT_0T_1 = \text{angle in alternate segment} = \tfrac{1}{2}\angle T_0OT_1 = \tfrac{1}{2}\theta.$$

Also $\angle AT_0T_2 = \frac{1}{2}\angle T_0OT_2 = \theta$, and $\angle AT_0T_3 = \frac{1}{2}\angle T_0OT_3 = \frac{3}{2}\theta$,

so that $$\angle AT_0T_1 = \angle T_1T_0T_2 = \angle T_2T_0T_3 = \tfrac{1}{2}\theta.$$

In the actual setting out a theodolite is placed at T_0, and taking AT_0 as the base line, the point T_1 is fixed, by means of a peg, one chain from T_0 in the direction T_0T_1 where $\angle AT_0T_1 = \frac{1}{2}\theta$, θ being previously calculated. The instrument is turned through the $\angle T_1T_0T_2 = \frac{1}{2}\theta$, and T_2 fixed one chain from T_1. Similarly, T_3 is fixed, and so on for the whole curve. It is clear that the value of θ is determined as the angle subtended by an arc of radius r standing on a chord one chain in length.

Ex. 63. *A circular curve is to be constructed of* 20·5 *chains radius, and it subtends an angle of* 22° 24′ *at the centre; find* (i) *the angle to be set off at the centre for each one-chain chord,* (ii) *the number of one-chain chords, and* (iii) *the length of the curve.*

(i) If in Fig. 53 we draw a perpendicular from O to T_1T_2, this perpendicular will bisect both T_1T_2 and the angle T_1OT_2;

$$\therefore \ \sin \tfrac{1}{2}\theta = \sin \tfrac{1}{2}T_1OT_2 = \tfrac{1}{2}/20{\cdot}5 = 1/41 = 0{\cdot}0244\,;$$

$\therefore$ from the tables, $\quad \tfrac{1}{2}\theta = 1° \, 24'$,

so that the angle subtended at the centre by a one-chain chord

$$= \theta = 2° \, 48'.$$

(ii) Since the total angle at centre $= 22° \, 24'$, and one chord subtends 2° 48′, therefore number of one-chain chords in the curve

$$= 22° \, 24'/2° \, 48' = 22{\cdot}4/2{\cdot}8 = \mathbf{8}.$$

(iii) Now $22° \, 24' = 0{\cdot}3910$ radian;

$$\therefore \ \text{length of arc} = 0{\cdot}391 \times 20{\cdot}5 = \mathbf{8{\cdot}02} \text{ chains}.$$

EXERCISES 8.

In each of the following examples the length of an arc and its radius are given; calculate the number of radians in the angle subtended at the centre by the arc.

1. Arc = 89·9 in., radius = 29 in.
2. Arc = 41·82 cm., radius = 17 cm.
3. Arc = 13·94 ft., radius = 16·4 ft.
4. Arc = 2381·4 cm., radius = 5·25 m.
5. Arc = 32·3 in., radius 15·2 in.
6. Arc = 80·36 yd., radius 117·6 ft.

Calculate the number of degrees and minutes in each of the following angles, which are given in radians.

7. 0·3491. 8. 0·6807. 9. 0·8203.
10. 1·11. 11. 1·302. 12. 2·892.

Calculate the number of radians in each of the following angles.

13. 14° 18′. 14. 38° 42′. 15. 45°.
16. 53° 24′. 17. 106° 48′. 18. 171° 36′.

In the following examples, r denotes the radius, l the length of an arc, and θ denotes the angle it subtends at the centre.

19. $r=2$ ft. 1 in., $\theta=0{\cdot}48$ radian, find l in feet.

20. $r=45$ yards, $\theta=40^\circ\ 20'$, find l in chains.

21. $r=186$ metres, $\theta=64^\circ\ 36'$, find l in kilometres.

22. $\theta=0{\cdot}76$ radian, $l=34{\cdot}2$ in., find r in yards.

23. $\theta=154^\circ\ 42'$, $l=33{\cdot}3$ yards, find r in feet.

24. $\theta=2{\cdot}75$ radians, $l=2$ miles, find r in yards.

25. Define a radian, and express in degrees and minutes an angle of 1·53 radians.

In December the sun is 91,400,000 miles distant from the earth, and the sun's diameter then subtends at the earth an angle of $32'\ 37''$; find the diameter of the sun correct to the nearest thousand miles. Find also the distance of the sun from the earth in June, given that the angle then subtended at the earth is $31'\ 34''$. (J.M.B.)

26. AP, BP are tangents at the extremities of a circular arc AB. The angle $APB=80^\circ\ 48'$ and $AP=47$ cm. Find (i) the radius of the arc, and (ii) the shortest distance between the arc and P.

27. The height of an arc standing on a chord 6·4 inches long is 1·8 inches; find (i) the radius of the arc, (ii) the angle it subtends at the centre, and (iii) the area of the segment bounded by the arc and chord.

28. The height of a circular arc standing on a chord 11·2 in. long is 3·2 in.; find (i) the radius of the arc, (ii) the angle subtended at the centre, and (iii) the areas of the sector and segment respectively.

29. Two places A and B, 3 miles apart in a straight line, are connected by a circular arc of railway of radius 10 miles. Find the length of the railway track. (D.S.)

30. The height of an arc standing on a chord 30 in. long is 9 in.; find (i) the radius of the arc, (ii) the angle subtended at the centre, (iii) the length of the arc.

31. A chord 8·3 inches long is placed in a circle of radius 9·8 in.; find (i) the angle subtended at the centre, (ii) the length of the arc, and (iii) the areas of the sector and segment respectively.

32. The height of an arc of radius 16·9 in. is 5 in.; find (i) the true length of the arc, (ii) the approximate length of the arc from Huyghens' formula, (iii) the percentage error in the approximate length.

33. The chord of an arc is 19·8 inches long and its height is 2 in.; find (i) the length of the arc by the approximate formula, (ii) the true length of the arc, and (iii) the percentage error in (i).

34. A circular arc of radius 2 ft. 5 in. is 8 in. high; find (i) the area of the segment formed by the approximate rule, and (ii) the true area of the segment.

35. The following formula is sometimes used to find the area A of the segment of a circle: $A=\frac{h^3}{2c}+\frac{2}{3}ch$, where h is the height and c the length of the chord of the segment. Obtain the value of c when $A=30$ and $h=4{\cdot}5$. (U.L.C.I.)

36. A circular hole, radius 12 inches, is partially closed by a circular disc whose radius is 13 inches. Find the approximate area of the crescent-shaped opening thus formed, if its central width is 4 inches. (U.L.C.I.)

37. The angle subtended by a tower on a plain at a very distant point is A°. At another point d yards further away from the tower, the angle subtended is B°. Shew that, if h be the height of the tower in feet,

$$\frac{\pi d}{60h}=\frac{1}{B}-\frac{1}{A}\text{ approximately.}$$ (J.M.B.)

38. At a curved part of a railway track, an engine going 15 miles per hour changes its direction of motion from N.E. to N. 24° E. in a quarter of a minute. At what rate is the direction changing in (i) radians per mile, (ii) radians per second? Assuming the curve to be an arc of a circle, find its radius in feet. (D.S.)

39. A uniform plate ABC is bounded by a circular arc AC of radius 4 ft. 5 in., and two straight edges AB, BC, such that AB is a tangent to the arc at A and angle ABC is 90°. Calculate the area of the plate in square inches when AB = 3 ft. 9 in., and BC = 2 ft. 1 in.

40. The perimeter of a sector of a circle is equal to that of half the circle; find the angle of the sector and its area, taking the radius as ten centimetres.

41. With the corner C of a square ABCD as centre, a circle is drawn cutting CD, CB in E, F respectively, such that it bisects the area of the square. Find its radius and the length of the chord EF if the side of the square is 5·2 inches long.

42. A curve of radius 47·75 chains is 45·01 chains long. Find how many one-chain chords can be set off in it, and the angle to be set out for each.

43. A railway running due North has to change its direction 43° E. of N., the radius of the curved portion being 528 ft. Find (i) the number of one-chain chords to be set out for the curve, and (ii) the length of the curve in yards to the nearest tenth.

CHAPTER IX

FUNCTIONS OF ANGLES GREATER THAN A RIGHT ANGLE

58. Angles greater than a Right Angle. So far we have been dealing only with acute angles. We must now consider how to deal with angles greater than a right angle.

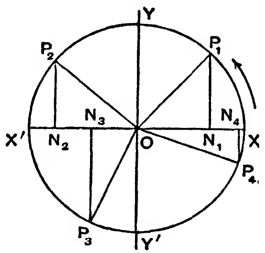


FIG. 54.

Ex. 64. *Calculate angles subtended at the centre of a circle by arcs greater in length than a quarter, a half, and three-quarters of the circumference respectively.*

On a piece of squared paper describe a circle with centre O and radius 6·5 units (Fig. 54). Draw two perpendicular diameters X′OX, Y′OY, parallel to the rulings. These divide the circle into four equal areas, XOY, YOX′, X′OY′, Y′OX, called the **first, second, third,** and **fourth quadrants** respectively.

Along OX cut off $ON_1 = 5{\cdot}6$ units and $ON_4 = 6{\cdot}3$ units; along OX′ cut off $ON_2 = 6$ units and $ON_3 = 5{\cdot}2$ units. Draw N_1P_1, N_2P_2 perpendicular to X′OX upwards meeting the circumference in P_1, P_2, and draw N_3P_3, N_4P_4 perpendicular to X′OX downwards meeting the circumference in P_3, P_4. Join OP_1, OP_2, OP_3, OP_4 and measure N_1P_1, N_2P_2, N_3P_3, N_4P_4. Now calculate each of the

acute angles XOP_1, P_2OX', $X'OP_3$, P_4OX from their tangents. Tabulate the results as follows :

$ON_1=5\cdot6$	$N_1P_1=3\cdot3$	$\tan XOP_1=N_1P_1/ON_1=0\cdot5892$	$\angle XOP_1=30°\,31'$
$ON_2=6$	$N_2P_2=2\cdot5$	$\tan P_2OX'=N_2P_2/N_2O=0\cdot4167$	$\angle P_2OX'=22°\,37'$
$ON_3=5\cdot2$	$N_3P_3=3\cdot9$	$\tan X'OP_3=P_3N_3/N_3O=0\cdot7500$	$\angle X'OP_3=36°\,52'$
$ON_4=6\cdot3$	$N_4P_4=1\cdot6$	$\tan P_4OX=P_4N_4/ON_4=0\cdot2540$	$\angle P_4OX=14°\,15'$

Now arc $XYP_2 > \frac{1}{4}$ circum., and angle at centre is $\angle XOP_2$,

,, $XP_2P_3 > \frac{1}{2}$,, ,, ,, ,, $\angle XOP_3$,

,, $XP_2P_4 > \frac{3}{4}$,, ,, ,, ,, $\angle XOP_4$,

the angles XOP_3, XOP_4 being measured in the direction of arrow.

To calculate these angles, we have, from Fig. 54, and the above table,

$$\angle XOP_2=180°-\angle P_2OX'=180°-22°\,37'=\mathbf{157°\,23'},$$

$$\angle XOP_3=180°+\angle X'OP_3=180°+36°\,52'=\mathbf{216°\,52'}, \text{ and}$$

$$\angle XOP_4=360°-\angle P_4OX=360°-14°\,15'=\mathbf{345°\,45'}.$$

59. Obtuse and Reflex Angles. Angles which are greater than one and less than two right angles are called **obtuse angles**, thus $\angle XOP_2$ is an obtuse angle, and it will be noticed that its arm OP_2 lies in the second quadrant; this will obviously be true of all obtuse angles.

When the sum of two angles is 180°, the angles are said to be **supplementary**, each being the **supplement** of the other. Thus the supplement of 57° is 123°, and the supplement of 162° is 18°, so that for every pair of supplementary angles, with the exception of two right angles, one is acute and the other obtuse.

Angles like XOP_3, XOP_4, each measured in the positive direction, *i.e.* in the direction of the arrow, which are greater than two right angles, are called **reflex** angles. To distinguish them the word ‘reflex’ is always written before the angle.

All reflex angles have one of their arms in either the third or fourth quadrants, so that we may say :

Acute angles belong to the **First** Quadrant.

Obtuse ,, ,, **Second** Quadrant.

Reflex ,, ,, **Third** and **Fourth** Quadrants.

EXERCISES 9A.

Draw diagrams showing the quadrant in which the revolving arm of each of the following angles lies. State also the kind of angle, and calculate the size of the acute angle in the same quadrant which must be added to each to make up a multiple of a right angle:

1. 30°.	2. 120°.	3. 225°.	4. 330°.
5. 46°.	6. 132°.	7. 231°.	8. 325°.

9. On a piece of squared paper describe a circle 8·5 cm. in radius whose centre is O. Draw a horizontal diameter X′OX and mark off X′A = 1 cm., X′B = 1·7 cm., X′C = 12·5 cm. and X′D = 16·2 cm. Erect perpendiculars AE, BF, CG, DH, meeting the circumference in E, F, G, H respectively, AE, DH being drawn downwards, and BF, CG upwards. Calculate (i) the lengths of these perpendiculars, (ii) the acute angles XOG, X′OE, X′OF, HOX, and (iii) ∠XOF, reflex ∠XOE, and reflex ∠XOH.

10. ABCD is a quadrilateral inscribed in a circle whose centre is O. If ∠ ABC = 67° 42′, calculate ∠ADC, ∠AOC and reflex ∠AOC.

11. ABC is a triangle in which AB = 12·3 in., BC = 15·5 in., CA = 21·8 in., and ∠ ABC is obtuse. Calculate the length of the perpendicular from A to CB produced, hence find the size of the angle ABC.

12. ABC is a triangle in which AB = 11·3 cm., BC = 16·5 cm., CA = 21·2 cm., and ∠ ABC is obtuse. The perpendicular from A to CB produced meets it in D; calculate the length of BD, and the size of ∠ABC.

13. ABCD is a rectangular sheet of paper; points P, Q are taken on AB, AD respectively such that AP = 17·7 cm. and AQ = 25 cm. The corner A is cut off along the line PQ; find the magnitudes of ∠ BPQ and ∠DQP.

Calculate the angle in degrees subtended at the centre of each of the following circular arcs, taking $\pi = 3{\cdot}1416$.

14. Length of arc = 20·25 in., radius = 7·5 in.

15. Length of arc = 84 cm., radius = 25 cm.

16. Length of arc = 82·35 yd., radius = 15 yd.

Calculate the size of the reflex angle in degrees between two adjacent sides of a regular polygon in each of the following cases where the number of sides is given:

17. 5 sides.	18. 8 sides.	19. 18 sides.
20. 25 sides.	21. 45 sides.	22. n-sides.

Find the number of sides of a regular polygon when the reflex angle between adjacent sides is

23. 190°.	24. 210°.	25. 220°

26 a right angles.

60. Positive and Negative Directions. In defining the trigonometrical functions of angles greater than a right angle, it is necessary to consider the directions in which the respective perpendiculars and bases are measured. All distances measured horizontally from left to right, and vertically upwards are taken as **positive**, whilst those measured in the opposite directions are taken as **negative**. For example, in the case of an obtuse angle such as XOP_2 (Fig. 54), the foot N_2 of the perpendicular N_2P_2 falls outside the angle, so that the base OX has to be produced backwards, *i.e.* in the negative direction. Hence ON_2 is negative. The same is true for reflex angles in the third quadrant, whilst the perpendiculars of all such angles have to be drawn downwards, and are therefore negative.

61. Trigonometrical Functions of Obtuse and Reflex Angles. We can now proceed to find the general values of the trigonometrical functions of angles greater than a right angle. Taking first any obtuse angle XOP_2 (Fig. 54), we have $ON_2 = -N_2O$, so that

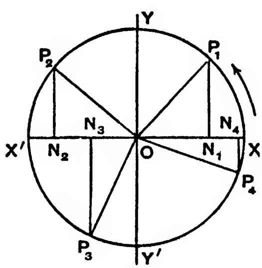


FIG. 54.

$$\tan XOP_2 = \frac{N_2P_2}{ON_2} = \frac{N_2P_2}{-N_2O} = -\tan P_2OX',$$

$$\cos XOP_2 = \frac{ON_2}{OP_2} = \frac{-N_2O}{OP_2} = -\cos P_2OX'.$$

$$\sin XOP_2 = \frac{N_2P_2}{OP_2} = \sin P_2OX'.$$

But $\angle P_2OX' = 180° - XOP_2$

$$\therefore \tan XOP_2 = -\tan(180° - XOP_2);$$
$$\cos XOP_2 = -\cos(180° - XOP_2);$$
$$\sin XOP_2 = \sin(180° - XOP_2).$$

Again, for the reflex angle XOP_3, $N_3P_3 = -P_3N_3$, so that

$$\tan XOP_3 = \frac{N_3P_3}{ON_3} = \frac{-P_3N_3}{-N_3O} = \frac{P_3N_3}{N_3O} = \tan X'OP_3,$$

$$\cos XOP_3 = \frac{ON_3}{OP_3} = \frac{-N_3O}{OP_3} = -\cos X'OP_3,$$

$$\sin XOP_3 = \frac{N_3P_3}{OP_3} = \frac{-P_3N_3}{OP_3} = -\sin X'OP_3.$$

But reflex $\angle XOP_3 = 180° + \angle X'OP_3$, from which,

$$\angle X'OP_3 = \angle XOP_3 - 180°,$$

$$\tan XOP_3 = \tan(XOP_3 - 180°);\ \cos XOP_3 = -\cos(XOP_3 - 180°);$$

$$\sin XOP_3 = -\sin(XOP_3 - 180°).$$

Finally, for a reflex angle in the fourth quadrant, such as reflex $\angle XOP_4$, we have $N_4P_4 = -P_4N_4$, so that

$$\tan XOP_4 = \frac{N_4P_4}{ON_4} = \frac{-P_4N_4}{ON_4} = -\tan P_4OX,$$

$$\cos XOP_4 = \frac{ON_4}{OP_4} = \cos P_4OX,$$

$$\sin XOP_4 = \frac{N_4P_4}{OP_4} = \frac{-P_4N_4}{OP_4} = -\sin P_4OX.$$

But reflex $\angle XOP_4 + \angle P_4OX = 360°$, from which

$$\angle P_4OX = 360° - \angle XOP_4;$$

$$\therefore\ \tan XOP_4 = -\tan(360° - XOP_4),\ \cos XOP_4 = \cos(360° - XOP_4),$$

$$\sin XOP_4 = -\sin(360° - XOP_4).$$

62. Summary of Results. The above results are so important that they are here collected in tabular form:

Trigonometrical Functions of angles greater than 90°.

When $\theta°$ is in the

Second Quadrant it is an Obtuse Angle lying between 90° and 180°; then	**Third Quadrant** it is a Reflex Angle lying between 180° and 270°; then	**Fourth Quadrant** it is a Reflex Angle lying between 270° and 360°: then
$\sin\theta = \sin(180° - \theta)$	$\sin\theta = -\sin(\theta - 180°)$	$\sin\theta = -\sin(360° - \theta)$
$\cos\theta = -\cos(180° - \theta)$	$\cos\theta = -\cos(\theta - 180°)$	$\cos\theta = \cos(360° - \theta)$
$\tan\theta = -\tan(180° - \theta)$	$\tan\theta = \tan(\theta - 180°)$	$\tan\theta = -\tan(360° - \theta)$

Ex. 65. *Find the trigonometrical functions of* 125°, 233°, *and* 327°. *What is the value of* cos 7A *when* A = 73°?

From the relations just established, since 125° is in the second quadrant,

$$\sin 125° = \sin(180° - 125°) = \sin 55° = \mathbf{0{\cdot}8192}.$$

$$\cos 125° = -\cos(180° - 125°) = -\cos 55° = \mathbf{-0{\cdot}5736}.$$

$$\tan 125° = -\tan(180° - 125°) = -\tan 55° = \mathbf{-1{\cdot}4281}.$$

Also, since 233° is in the third Quadrant,

$$\sin 233° = -\sin(233° - 180°) = -\sin 53° = -0{\cdot}7986.$$
$$\cos 233° = -\cos(233 - 180°) = -\cos 53° = -0{\cdot}6018.$$
$$\tan 233° = \tan(233° - 180°) = \tan 53° = 1{\cdot}3270.$$

Finally, since 327° is in the Fourth Quadrant,

$$\sin 327° = -\sin(360° - 327°) = -\sin 33° = -0{\cdot}5446.$$
$$\cos 327° = \cos(360° - 327°) = \cos 33° = 0{\cdot}8387.$$
$$\tan 327° = -\tan(360° - 327°) = -\tan 33° = -0{\cdot}6494.$$

When A = 75°, 7A = 511°, which is greater than four right angles. But 511° = 360° + 151°, so that the revolving arm has traversed a complete revolution and 151° in addition, *i.e.* 511° is effectively equal to 151°;

$$\therefore \cos 511° = \cos 151° = -\cos(180° - 151°) = -\cos 29°$$
$$= -0{\cdot}8746.$$

63. Variation of sin θ as θ increases from 0° to 360°. We must now investigate how sin θ changes as θ increases from 0° to 360°. This is best done graphically.

PROBLEM 16. *To trace the variation of sin θ as θ changes from* 0° *to* 360°.

On a piece of squared paper describe, near the left-hand edge, a circle of radius 5 cm., marking H as its centre (Fig. 55). Draw

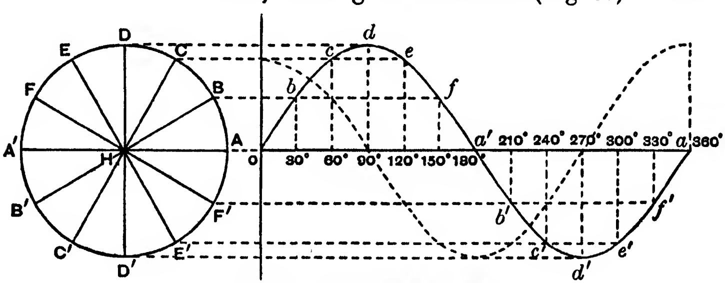


FIG. 55.—Graphs of sin θ and cos θ.

a diameter A'HA and produce it completely across the paper. Divide the upper semicircle into six equal parts at B, C, D, E, F, and draw the diameters BHB', CHC', DHD', EHE' and FHF'; then ∠AHB = 30°, ∠AHC = 60°, ∠AHD = 90°, and so on. To represent these angles, take a point O on A'A produced and mark off to the

right twelve equal parts, and number them 30°, 60°, 90°, etc., as shewn in the figure. From B, draw a line parallel to A′A to meet the ordinate at 30° in b. Do this for each of the other points on the circle, thus obtaining a series of corresponding points b, c, d, e, ... a. Through these draw a smooth curve.

Now sin AHB = (perp. on HA from B)/HB = (ordinate at 30°)/5,

sin AHC = (perp. on HA from C)/HC = (ordinate at 60°)/5,

and so on for each of the other angles; hence, since the radius is the same for all, the sine of each angle is proportional to its ordinate, and therefore the curve shews the changes in the value of $\sin\theta$ as θ varies from 0° to 360°.

It should be noticed that

(i) from 0° to 180°, the sine is positive, whilst from 180° to 360° it is negative,

(ii) the sine of any acute angle = the sine of its supplement.

(iii) $\sin(180° + \alpha) = \sin(360° - \alpha°)$, where α is an acute angle,

(iv) the curve repeats itself for values of θ beyond 360°; for this reason the sine is said to be a **periodic** function with a period of 4 right angles or 2π radians,

(v) the greatest value of $\sin\theta$ occurs at 90°, where it is 1, and the least value at 270°, where it is -1.

64. Variation of $\cos\theta$ as θ increases from 0° to 360°. The changes in the value of $\cos\theta$ as θ varies from 0° to 360° may also be shewn on the same figure by marking off on each ordinate a distance equal to the base of the corresponding angle, *i.e.* the segment of HA (Fig. 55) cut off by the perpendicular from the extremity of the corresponding radius. By joining the extremities of the ordinates thus marked off, we get another curve precisely similar to that of the sine. Indeed, it will be seen from the figure that it is really the sine curve moved horizontally through 90° to the left. Hence for any angle θ,

$$\cos\theta° = \sin(\theta° + 90°).$$

We also see that **the value of $\cos\theta$, like that of $\sin\theta$, lies between $+1$ and -1**, and

$$\sin 0° = \sin 180° = \sin 360° = 0,$$
$$\sin 90° = -\sin 270° = 1,$$
$$\cos 0° = -\cos 180° = \cos 360° = 1,$$
$$\cos 90° = \cos 270° = 0.$$

65. Variation of tan θ as θ increases from 0° to 360°. Since the tangent of an angle is the ratio of the perpendicular to the base, by making the base the same for all angles from 0° to 360°, the tangent will be proportional to the perpendicular. The variation in the length of the perpendicular will therefore give the corresponding variation in the tangent.

PROBLEM 17. *To trace the changes in the values of tan θ as θ varies from* 0° *to* 360°.

On a piece of squared paper draw a circle 5 cm. in radius having

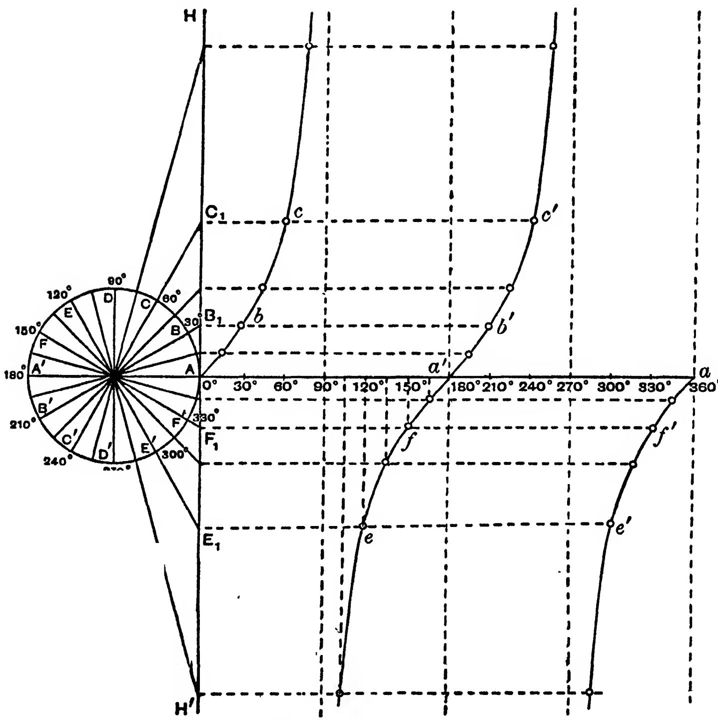


FIG. 56.—Graph of tan θ.

A′A (Fig. 56) as diameter. Draw H′H through A perpendicular to A′A. Divide the circumference into twelve equal parts and draw

diameters through the points of division producing them to meet H′H as indicated. Draw lines parallel to the axis Aa from each of the points on H′H, to intersect the corresponding ordinates. A series of points is thus obtained through which smooth curves, as shewn, may be drawn.

66. Characteristics of the Tangent Curves. From Fig. 56, it will be noticed that the tangent curve is not continuous like that of the sine or cosine. It consists of three portions, breaks occurring at 90° and at 270°, so that no finite values are given for the tangents of these angles. As, however, the angle increases from 0° to 90° the intercept on AH gets longer and longer, until at 90°, where D′D is parallel to AH, it is greater than any finite line. Look up a table of tangents from 85° to 89° 54′ and see how rapidly they increase from 11·43 to 573. From 89° 54′ to 89° 59′ the values are :

θ	$\tan\theta$	θ	$\tan\theta$
89° 54′	573·0	89° 57′	1145·9
55′	687·5	58′	1718·9
56′	859·4	59′	3437·7

It is evident, therefore, that we can make the tangent of an acute angle as large as we please by making the angle near enough to 90°, *i.e.* tan 90° is greater than any finite number. This is expressed by the symbolical statement

$$\tan 90° = \infty,$$

where the symbol ∞ is read "infinity".

Similar facts apply to tan 270°, so that we may write

$$\tan 270° = \infty .$$

It will be observed also that as θ passes through 90°, $\tan\theta$ changes from $+\infty$ to $-\infty$, and as θ passes through 270°. $\tan\theta$ changes from $-\infty$ to $+\infty$. Further, $\tan 0° = 0$, $\tan 180° = 0$ and $\tan 360° = 0$, so that **$\tan\theta$ may have any value whatever between $-\infty$ and $+\infty$**; *i.e.* an angle can always be found having for its tangent any number, positive or negative, however great or small.

EXERCISES 9B.

Find the sine, cosine and tangent of each of the following angles:

1 $152°$. 2. $138° 42'$. 3. $154° 36'$. 4. $196°$.

5. $215° 30'$. 6. $282°$. 7. $315° 18'$. 8. $329° 24'$.

9. $371°$. 10. $492°$. 11. $540°$. 12. $621°$.

13. The electro-motive force (E.M.F.) in a certain alternating current circuit is equal to $100 \sin \theta$. What are the values of the E.M.F.'s for the following values of θ: $40°$, $145°$, $235°$, $310°$, and what is the value of θ when the E.M.F. is 90 volts? (U.L.C.I.)

Verify each of the following statements:

14. $\sin 153° = \sin 77° \cos 76° + \sin 76° \cos 77°$.

15. $\cos 142° = \cos 74° \cos 68° - \sin 74° \sin 68°$.

16. $\tan 178° = \dfrac{\tan 81° + \tan 97°}{1 - \tan 81° \tan 97°}$.

17. $\sin 236° = 2 \sin 118° \cos 118°$.

18. $\cos 242° = \cos^2 121° - \sin^2 121°$.

19. $\tan 212° = \dfrac{2 . \tan 106°}{1 - \tan^2 106°}$.

20. $\sin 135° + \sin 87° = 2 . \sin 111° . \cos 24°$.

21. $\cos 147° + \cos 79° = 2 . \cos 113° . \cos 34°$.

22. Evaluate $\sin A + \sin 2A + \sin 3A + \sin 4A$ when $A = 72°$.

23. Find the value of $\dfrac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A . \cos 2A}$, correctly to two places of decimals, when $A = 82°$.

24. If $\tan \theta = \dfrac{\sin A + \sin 2A + \sin 3A + \sin 4A}{\cos A + \cos 2A + \cos 3A + \cos 4A}$, find the smallest positive value of θ when $A = 82°$.

25. If $\tan \theta = \dfrac{\sin 3A + 2 \sin 5A + \sin 7A}{\cos 3A + 2 \cos 5A + \cos 7A}$, find the smallest positi value of θ when $A = 43°$.

Find the positive values of θ, between $0°$ and $360°$, which satisfy each of the following relations:

26. $\sin \theta = 0{\cdot}512$. 27. $\sin \theta = 0{\cdot}91$. 28. $\cos \theta = 0{\cdot}682$.

29. $\cos \theta = 0{\cdot}4019$. 30. $\tan \theta = 0{\cdot}4$. 31. $\tan \theta = 13$.

32. $\sin \theta = -0{\cdot}304$. 33. $\cos \theta = -0{\cdot}81$. 34. $\tan \theta = -1$.

35. $\sin\theta = \frac{1}{4}(\sqrt{6} - \sqrt{2})$.

36. $\cos\theta = \frac{1}{4}(\sqrt{5} + 1)$.

37. $\tan\theta = 2 + \sqrt{3}$.

38. $\cos\theta = -\frac{1}{4}(\sqrt{6} + \sqrt{2})$.

39. $\sin\theta = \frac{1}{4}(1 - \sqrt{5})$.

40. $\tan\theta = -\sqrt{5 + 2\sqrt{5}}$.

41. Draw the graph of $\cot\theta$ between $0°$ and $360°$. Deduce the values of the cotangent when $\theta = 0°, 90°, 180°, 270°$.

42. Draw the graph of $y = \cos\left(\theta + \frac{\pi}{6}\right) - \cos\left(\theta + \frac{\pi}{2}\right)$ from the point at which $\theta = 0$ to the point at which $\theta = 2\pi$. (J.M.B.)

43. Draw the graph of $y = \sin\theta + \cos\theta$ from $\theta = 0$ to $\theta = 180°$, and from it read off the value of y when $\theta = 21°$.

44. Draw the graph of $y = \tan\theta + \cot\theta$ from $\theta = 0$ to $\theta = 180°$; hence find the value of y when $\theta = 70{\cdot}3°$. Shew also from the graph that the sum of a number and its reciprocal cannot be less than 2.

45. Plot the function $y = 6{\cdot}8\sin\theta + 100\cos\theta$ on squared paper from $\theta = 0$ to $\theta = 90°$, and read off the value of θ when $y = 90$.

46. Draw the graph of $y = \tan\theta + 13\cot\theta$ from $\theta = 0$ to $\theta = 90°$, and read off the values of θ when $y = 14$.

CHAPTER X

THE SECANT AND COSECANT. REGULAR POLYGONS

67. Further Conversion Factors. We know from Art. 36 that the trigonometrical functions are convenient multipliers for converting one side of a right-angled triangle into another side. It will be observed, however, that there are no functions which, as multipliers, will convert either of the sides forming the right angle into the hypotenuse. The following example will shew this.

Ex. 66. *A tie wire of a vertical pole is attached to the pole at a point* 8·3 *ft. from the top, and to the ground at a distance* 4·5 *ft. from the foot of the pole. The angle made by the wire with the ground, which is horizontal, is* 77° ; *find the lengths of the wire and of the pole.*

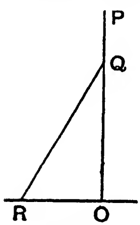


FIG. 57.

Let OP (Fig. 57) be the pole, RQ the wire, then OR = 4·5 ft., PQ = 8·3 ft., and $\angle$ ORQ = 77°.

Since ORQ is a right-angled triangle,

$$\therefore \text{OR} = \text{RQ} \cos 77°, \quad i.e.\ 4{\cdot}5 = \text{RQ} \times 0{\cdot}225,$$

so that $$\text{RQ} = 4{\cdot}5/0{\cdot}225 = 20 \text{ ft.}$$

Also

$$\text{OQ} = \text{RQ} \sin 77° = 20 \times 0{\cdot}9744 = 19{\cdot}488;$$

$$\therefore \text{OP} = \text{OQ} + \text{QP} = 19{\cdot}488 + 8{\cdot}3 = \mathbf{27{\cdot}8} \text{ ft.}$$

68. The Secant and Cosecant. In such calculations as Ex. 66, it would be more convenient if we had simple multipliers to convert either of the sides OR, OQ respectively into the hypotenuse RQ. Suppose $\angle$ ORQ (Fig. 57) be any angle θ, then RQ is equal either to OR/$\cos\theta$ or to OQ/$\sin\theta$.

Now $1/\cos\theta$ is the ratio of the hypotenuse to the base, and is called the **secant** of θ; similarly, $1/\sin\theta$ is the ratio of the hypo-

tenuse to the perpendicular, and is called the **cosecant** of θ. The secant and cosecant of θ are briefly written sec θ and cosec θ respectively, so that

$$\sec\theta = \frac{1}{\cos\theta} \quad \text{and} \quad \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

We have now all the elementary trigonometrical functions, viz. sin θ, cos θ, tan θ, and their respective reciprocals; cosec θ, sec θ, cot θ. With the aid of these, any one side of a right-angled triangle may be converted into either of the other two by simple multiplication, when we have the requisite tables of the six functions.

Ex. 67. *Given that tan* A $=4{\cdot}95$, *find the other five trigonometrical functions of* A *without using tables, and shew that*

$$\cot A = 10\,(\operatorname{cosec} A - 1).$$

$$\tan A = 4{\cdot}95 = \tfrac{495}{100} = \tfrac{99}{20}.$$

Now draw a triangle ABC, having a right angle at C, BC $=99$ units and CA $=20$ units; then tan A $=$ BC/CA $=99/20$.

If, therefore, we know the length of the hypotenuse AB, all the functions of A can be written down. By the theorem of Pythagoras,

$$AB^2 = BC^2 + CA^2 = 99^2 + 20^2 = 9801 + 400 = 10201;$$

$$\therefore\ AB = \sqrt{10201} = 101.$$

Hence

$$\begin{aligned}
\sin A &= BC/AB = 99/101 = \mathbf{0{\cdot}9802}.\\
\cos A &= CA/AB = 20/101 = \mathbf{0{\cdot}1980}.\\
\sec A &= AB/CA = 101/20 = \mathbf{5{\cdot}05}.\\
\operatorname{cosec} A &= AB/BC = 101/99 = \mathbf{1{\cdot}0202}.\\
\cot A &= CA/BC = 20/99 = \mathbf{0{\cdot}2020}.
\end{aligned}$$

Taking the right-angled side of the given relation

$$10(\operatorname{cosec} A - 1) = 10(\tfrac{101}{99} - 1) = 20/99 = \cot$$

EXERCISES 10A.

Find cosec A as a decimal, without using tables, when sin A has each of the following values:

1. $\frac{1}{2}$. 2. 117/125. 3. 0·81.

4. 0·319. 5. $\sqrt{3}/2$. 6. $(\sqrt{6}-\sqrt{2})/4$.

Find sec A as a decimal, without using tables, when cos A has each of the following values:

7. 3/4. 8. 7/25. 9. 0·91.

10. 0·565. 11. $\sqrt{2}/2$. 12. $(\sqrt{5}-1)/4$.

Find cot A as a decimal, without using tables, when tan A has each of the following values:

13. 2/5. 14. 225/216. 15. 10·2.

16. 35·8. 17. $\sqrt{3}$. 18. $2-\sqrt{3}$.

Calculate the five other trigonometrical functions as decimals when:

19. $\tan A = 13/84$. 20. $\cot A = 39/80$. 21. $\sin A = 0{\cdot}96$.

22. $\cos A = 0{\cdot}504$. 23. $\sec A = 2$. 24. $\operatorname{cosec} A = 1{\cdot}45$.

Using the tables, verify each of the following relations:

25. $\sec 26^\circ\ 12' = \tan 48^\circ\ 6'$.

26 $\sec 68^\circ\ 48' = \operatorname{cosec}(90^\circ - 68^\circ\ 48')$.

27. $\operatorname{cosec} 11^\circ\ 30' = \sec 11^\circ\ 30' \,.\, \cot 11^\circ\ 30'$.

28. $\sin 30^\circ\ 24' \,.\, \operatorname{cosec} 30^\circ\ 24' = 1$.

29. $\cos 72^\circ \,.\, \sec 72^\circ = 1$.

30. $\tan 84^\circ\ 54' \,.\, \cot 84^\circ\ 54' = 1$.

In a triangle ABC, if

31. $\Delta = 394$ sq. in., $A = 52^\circ$, $b = 25$ in., find c.

32. $\Delta = 4575$ sq. cm., $B = 66^\circ\ 12'$, $c = 125$ cm., find a.

33. $\Delta = 74{\cdot}9$ sq. ft., $C = 48^\circ\ 30'$, $a = 16$ ft., find b.

34. $\Delta = 60{\cdot}72$ sq. yd., $A = 30^\circ\ 24'$, $b = 15$ yd., find c.

Solve the following triangles:

35. $a = 6{\cdot}5$ in., $B = 48^\circ$, $C = 90^\circ$. 36. $b = 25$ ft., $C = 90^\circ$, $A = 52^\circ$.

37. $a = 31$ in., $A = 61^\circ$, $C = 90^\circ$. 38. $b = 42$ ft., $B = 72^\circ$, $C = 90^\circ$.

39. $a = 12{\cdot}8$ yd., $B = C = 30^\circ\ 42'$. 40. $a = 35$ chains, $B = C$, $A = 43^\circ$.

41. $b = 86$ yd., $B = 38^\circ$, $C = A$. 42. $b = 72$ cm., $C = A = 73^\circ$.

43. $c = 94$ yd., $A = B = 62^\circ$. 44. $c = 112$ cm., $A = B$, $C = 84^\circ$.

45. $B = 49^\circ$, $C = 81^\circ$, and the perpendicular to BC from $A = 28$ m.

46. $A = 45^\circ$, $B = 115^\circ\ 24'$, and the perpendicular from C to AB produced, meeting it in D, makes $AD = 15$ m.

69. Circles and Perimeter of a Regular Polygon. A regular polygon can always be described in a circle, so that each side is a chord. This circle is called the **circumscribed** circle, or more briefly, the **circum-circle.** It is also possible to describe a concentric circle to touch each side of the polygon. Such a circle is called the **inscribed** or **in-circle.** If the side of the polygon be given, we can easily find the radii of the circum- and in-circles, and conversely, if the radius of either circle be known the side of the polygon may readily be found.

Ex. 68. *Find the radii of the circum- and in-circles of a regular nonagon each of whose sides is* 5 *cm.*

Let AB (Fig. 58) be one side of the nonagon, D its mid-point and O the centre of the circum- and in-circles; then OD is perpendicular to AB, and the inscribed circle touches AB at D.

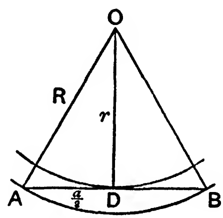


Fig. 58.

Since the polygon has nine sides, by joining O to the angular points, there will be nine equal isosceles triangles of which AOB is one;

$$\therefore \angle AOB = \tfrac{1}{9}.(4 \text{ right angles}) = 40^\circ,$$

and $$\angle AOD = \angle DOB = \tfrac{1}{2}\angle AOB = 20^\circ.$$

Hence the radius AO of the circum-circle $=$ AD cosec AOD

$$= 2{\cdot}5 \operatorname{cosec} 20^\circ = 2{\cdot}5 \times 2{\cdot}9238 = 7{\cdot}3095 \text{ cm.}$$
$$= \mathbf{7{\cdot}3} \text{ cm. practically.}$$

Again, the radius of the in-circle

$$= OD = AD \cot AOD = 2{\cdot}5 \cot 20^\circ = 2{\cdot}5 \times 2{\cdot}7475 = 6{\cdot}8687 \text{ or } \mathbf{6{\cdot}9} \text{ cm.}$$

We might have found this radius thus:

$$DO = OA \cos 20^\circ = 7{\cdot}3 \times 0{\cdot}9397 = 6{\cdot}9 \text{ cm.}$$

Ex. 69. *The radius of a circle is five inches, and a regular polygon of* 15 *sides is inscribed in it. Find the perimeter of the polygon and the angle between the two adjacent sides.*

Referring again to Fig. 58, let AB represent one of the fifteen sides, then $\angle AOB = 360°/15 = 24°$;

$$\therefore AB = 2AD = 2AO \cdot \sin AOD = 2{\cdot}5 \cdot \sin 12°$$
$$= 2{\cdot}5 \times 0{\cdot}2079 = 2{\cdot}079 \text{ in.};$$
$$\therefore \text{Perimeter} = 2{\cdot}079 \times 15 = 31{\cdot}185 \text{ in. or } \mathbf{31{\cdot}2} \text{ in.}$$

Now $\angle OAD$ = half the angle between two adjacent sides

$$= 90° - 12° = 78°;$$
$$\therefore \text{Angle of polygon } 78° \times 2 = \mathbf{156°}.$$

70. General Rule. General expressions will now be found connecting the side and the circum- and in-radii of a regular polygon.

Problem 18. *A regular polygon of n sides is described in a circle of radius* R. *If a be the length of a side and r the radius of the in-circle, find the relations between a, r and* R.

Let AB (Fig. 58) be a side of the polygon, O the centre of the circum-circle, and OD the perpendicular on AB from O; then $AD = \frac{1}{2}AB = \frac{1}{2}a$, $OA = R$, and $OD = r$.

∴ By the theorem of Pythagoras,

$$\mathbf{R^2 = \tfrac{1}{4}a^2 + r^2}.$$

Again, $\angle AOD = \frac{1}{2}\angle AOB = \frac{1}{2} \cdot \frac{2\pi}{n} = \frac{\pi}{n}$ radians.

But $DO = AO \cos AOD$, or $AD \cot AOD$,

i.e.
$$\mathbf{r = R \cos \frac{\pi}{n}} \quad \text{or} \quad \mathbf{r = \tfrac{1}{2}a \cot \frac{\pi}{n}}.$$

Finally, $AD = AO \sin AOD$, *i.e.*

$$\mathbf{a = 2R \sin \frac{\pi}{n}}.$$

71. Expressions for the Area of a Regular Polygon. We have already seen how to find the area of a regular polygon (p. 74), and now we shall deduce general expressions for the area in terms of a, r, R and n.

PROBLEM 19. *A regular polygon of n sides is described in a circle of radius* R. *If a be a side and r the in-radius, find expressions for its area in terms of a, r, n and* R.

From Fig. 58, it is clear that $\triangle OAB = \frac{1}{2} . AB . DO = \frac{1}{2}ar$.

Now, since there are n triangles in the polygon each equal to OAB, we have, using the results of Problem 18,

$$\text{Area of polygon} = \tfrac{1}{2}nar = \tfrac{1}{2}naR\cos\frac{\pi}{n} = \tfrac{1}{4}na^2\cot\frac{\pi}{n} = nr^2\tan\frac{\pi}{n}$$

$$= nR^2\sin\frac{\pi}{n} . \cos\frac{\pi}{n}.$$

Ex. 70. *A flat plate in the shape of a regular octagon is cut from a circular sheet of diameter 3 ft. 6 in. with the minimum waste. Find the length of a side and the area of the plate.*

A regular octagon is a polygon of 8 sides; hence if a inches be the length of one side, we have from Prob. 19,

$$a = 2R\sin(180°/8) = 2R\sin 22{\cdot}5°.$$

Now since the waste is to be the minimum, the circular plate will correspond to the circum-circle of the octagon;

$$\therefore\ 2R = 3{\cdot}5 \text{ ft., so that } R = 1{\cdot}75 \text{ ft.}$$

$$\therefore\ a = 3{\cdot}5\sin 22{\cdot}5° = 3{\cdot}5 \times 0{\cdot}3827 = \mathbf{1{\cdot}34} \text{ ft.}$$

Further, from Prob. 19, the area of the plate

$$= 8 \times 1{\cdot}75^2 \sin 22{\cdot}5° \cos 22{\cdot}5°$$

$$= 8 \times 1{\cdot}75^2 \times 0{\cdot}3827 \times 0{\cdot}9239 = \mathbf{8{\cdot}66} \text{ sq. ft.}$$

72. Area-multipliers. Often in practice the only known measurement of a regular polygon is either the length of a side or that of the circum-radius. Now from Problem 19, the area of a polygon is either $\frac{1}{4}n\cot(\pi/n) . a^2$, or $n\sin(\pi/n)\cos(\pi/n) . R^2$, so that $\frac{1}{4}n\cot(\pi/n)$ and $n\sin(\pi/n)\cos(\pi/n)$ are multipliers which will convert the square of a side and the square of the circum-radius respectively into the area.

For those polygons most frequently met with, these area-multipliers are therefore calculated and tabulated for reference. A table of them will be found at the end of the book on p. 205.

Ex. 71. *Solve* Ex. 70 *again by the use of a table of area-multipliers.*

For a regular octagon of side a, and circum-radius R, its area is, from the table on p. 205,

$$4{\cdot}828a^2 \quad \text{or} \quad 2{\cdot}828\mathsf{R}^2.$$

Since $\mathsf{R}=1{\cdot}75$ ft.; $\therefore$ area $=2{\cdot}828\times1{\cdot}75^2=\mathbf{8{\cdot}66}$ sq. ft.

Also, $$4{\cdot}828a^2=2{\cdot}828\mathsf{R}^2;$$

$$\therefore\ a=1{\cdot}75\sqrt{2{\cdot}828/4{\cdot}828}=\mathbf{1{\cdot}34}\text{ ft.}$$

EXERCISES 10B.

Angles of Quadrilaterals and Polygons.

1. A certain polygon has seven sides; one of the angles is 25° greater than each of the others, which are equal. Find the number of degrees in each angle. (L.M.)

2. ABCDEF is a hexagon in which $\angle$FAB=110°, $\angle$ABC=115°, the side CD is at right angles to side AB, and the side DE is at right angles to side AF. The angles BEF, EFA are equal. Find all the angles of the hexagon other than those given. (L.S.)

3. The sum of the interior angles of a regular polygon is four times the sum of the exterior angles formed by producing the sides successively; find the number of sides in the polygon.

4. In the hexagon ABCDEF, each of the sides AB, BC, CD, DE is one inch long; the angles at A, C, E are equal to one another, as also are the angles at B, D, F, the former being double the latter. Find these angles, draw the hexagon, and measure the lengths AF, AD. (L.M.)

5. The angles A, B, C of a quadrilateral ABCD are 96°, 140°, 76° respectively, and the bisectors of the four exterior angles of the quadrilateral are drawn, forming a quadrilateral PQRS; find the angles of this quadrilateral. (L.M.)

6. ABCD is a cyclic quadrilateral such that BA, CD produced intersect at an angle of 30°, and DA, CB produced intersect at an angle of 40°. Find the angles of the quadrilateral. (L.M.)

7. In a pentagon ABCDE, the sides AB, BC, EA are each one inch in length, the angles A, B are each 120°, and the angles C, E are each 95°. Alternate sides are produced to meet, forming a star-shaped figure. Calculate the magnitudes of the angles of the star. (L.M.)

8. ABCDE is an irregular pentagon having the angle B equal to the angle D, each being equal to two-thirds of the sum of the angles A and C. The angle A is double the angle C, and E is three times C. Find the size of each of the angles.

9. If ABCDEFG is a regular heptagon, calculate to the nearest minute the angles of the figure, the angles of the triangle ACF, and the angles of the triangle formed by AC, DF and the perpendicular at G to FG. (L.M.)

10. ABCDE is a convex pentagon with the side BC equal and parallel to AE, and the three sides AB, CD, DE all equal to one another. If the angle at A is 70°, find each of the other angles of the pentagon. (L.M.)

11. In a cyclic quadrilateral ABCD, the angles CAD, BDA, BDC are 15°, 65°, 35° respectively. Find the angles of the quadrilateral, and construct it if AC $=1{\cdot}7$ in., measuring the length of BD. (L.M.)

12. A square ABCD and a regular pentagon ABPQR are described on opposite sides of a line AB. By calculating the sizes of the angles RAD, PRD, prove that one of these angles is double the other. (L.S.)

13. The magnitudes of the angles of a polygon, taken in order round the figure, are in arithmetical progression. The smallest is 107° and the greatest is 163°; find the number of sides in the figure.

14. PSQ is a circle of radius r and centre O, having POQ as a diameter, and OS perpendicular to PQ. R is the mid-point of OP, and with R as centre and RS as radius, an arc is drawn cutting OQ in T. Shew that $2\,.\,\mathrm{ST}^2=(5-\sqrt{5})r^2$; hence prove that ST is the length of a side of the inscribed pentagon.

Areas of Polygons.

The expressions given in Prob. 19 for the area of a regular polygon may be written in the forms, k_1a^2, $k_2\mathrm{R}^2$, k_3r^2, where k_1, k_2, k_3 are constants for each polygon. Calculate the values of these multipliers for each of the following regular polygons:

15. Pentagon. 16. Hexagon. 17. Heptagon.

18. Octagon. 19. Decagon. 20. Duodecagon.

Find the areas of each of the following regular polygons:

21. A decagon of side 8·5 in.

22. A heptagon whose perimeter is 4 ft. $4\frac{1}{2}$ in.

23. A polygon of 15 sides inscribed in a circle of 25·6 in. diameter.

24. A polygon of 18 sides inscribed in a circle of radius 18·3 cm.

25. A polygon of 24 sides circumscribed about a circle of 17·4 in. diameter.

26. A polygon of 30 sides circumscribed about a circle of 15·2 cm. diameter.

27. An octagon formed by cutting off the corners of a square of 2 ft. 11 in. side. Give the area in square feet.

28. A regular hexagonal plate is to be made by cutting off the corners of a rectangular plate 2 ft. 1 in. long. How wide must the plate be, and what will be the area of the hexagon?

29. A regular octagonal court has a perimeter of 176 yards; find its area in acres.

30. The area of a regular duodecagon is 1587 sq. cm. Find the radius of the circumscribed circle.

31. In any regular polygon, shew that if s = semi-perimeter, and A = area, then the radius of the in-circle $= A/s$; hence find the length of one side of a nonagon of $r = 6{\cdot}5$ in. and $A = 81{\cdot}9$ sq. in.

32. Prove that the areas of regular polygons, each of n sides, inscribed in and circumscribed about a given circle are as

$$\cos^2(180°/n) : 1.$$

If this ratio be 0·75, find n.

33. ABCDE is a regular pentagon whose perimeter is 3 ft. 7 in. The sides AB, DC are produced to meet in F; find the area of the triangle BCF.

34. A regular hexagon with side 4 in. and a concentric circle of the same area as the hexagon are described. Find the length of the chord cut off by the circle from each side of the hexagon. (L.M.)

CHAPTER XI

SIMPLE SOLUTION OF A TRIANGLE

73. Data for Solution. We shall now consider briefly the simple methods of solving a triangle completely. There are four cases to be dealt with, viz., those in which we are given:

(i) Three sides,
(ii) Two sides and the included angle,
(iii) One side and two angles,
(iv) Two sides and the angle opposite one of them.

74. Case I. When Three Sides are given. Here we have to find the three angles.

Ex. 72. *Solve the triangle in which* $a=89$ *cm.*, $b=97$ *cm. and* $c=170$ *cm.*

Let ABC (Fig. 59) be the triangle; then clearly A and B will be acute angles, since they are opposite the shorter sides. Draw CF perpendicular to AB, then we must determine the lengths of AF and FB.

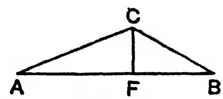


FIG. 59.

Let AF $=x$ cm., so that FB $=170-x$ cm.; then

$$AC^2-AF^2=CF^2=CB^2-FB^2,$$

or

$$97^2-x^2=89^2-(170-x)^2,$$

from which $\quad x=7597/85 \quad$ and $\quad 170-x=6853/85$;

$$\therefore\ \cos A=AF/AC=7597/(85\times 97)=0{\cdot}9212,$$

and

$$\cos B=FB/BC=6853/(85\times 89)=0{\cdot}9059.$$

Hence, from the table of cosines,

$$A=22^\circ\ 54' \quad\text{and}\quad B=25^\circ\ 3';$$

$$\therefore\ C=180^\circ-(A+B)=180^\circ-47^\circ\ 57'=132^\circ\ 3'.$$

The triangle is thus completely solved.

EXERCISES 11A.

Solve completely each of the following triangles:

1. $a=4$ in., $b=5$ in., $c=6$ in.
2. $a=7$ ft, $b=6$ ft., $c=5$ ft.
3. $a=13$ yd., $b=12$ yd., $c=5$ yd.
4. $a=12$ cm., $b=37$ cm., $c=35$ cm.
5. $a=1+\sqrt{3}$, $b=\sqrt{6}$, $c=2$.
6. $a=6$, $b=2\sqrt{6}$, $c=3\sqrt{2}+\sqrt{6}$.
7. $a=56$, $b=c=53$.
8. $a=92$, $b=85$, $c=39$.
9. $a=b=73$, $c=96$.
10. $a=73$, $b=52$, $c=75$.

75. The Cosine Rule. To save the trouble of finding the segments into which the perpendicular divides the base in every case, we can find a general rule which will give the cosine at once.

PROBLEM 20. *To find a relation connecting the three sides of a triangle with one of its angles.*

Let ABC be the triangle (Fig. 59), and let F be the foot of the perpendicular from C; let $AF=x$, then $FB=c-x$;

$\therefore$ as in Ex. 72, $\quad b^2-x^2=a^2-(c-x)^2$,

from which $\quad b^2-a^2+c^2=2cx$.

Now $\quad x=AF=AC\cos A=b\cos A$;

$$\therefore\ b^2-a^2+c^2=2bc\cos A,$$

or

$$\mathbf{a^2=b^2+c^2-2bc\cos A.}$$

Similarly, it may be shewn that

$$\mathbf{b^2=c^2+a^2-2ca\cos B} \quad \text{and} \quad \mathbf{c^2=a^2+b^2-2ab\cos C.}$$

These important relations constitute what is called the **Cosine Rule** of a triangle.

In calculating angles by this rule it is convenient to modify the expression for the cosine in the following way.

Since $a^2=b^2+c^2-2bc\cos A$; $\therefore$ $\cos A=(b^2+c^2-a^2)/2bc$, so that

$$\begin{aligned} 1+\cos A &= (b^2+c^2-a^2+2bc)/2bc \\ &= \{(b+c)^2-a^2\}/2bc \\ &= (b+c+a)(b+c-a)/2bc \\ &= 2s(s-a)/bc, \end{aligned}$$

where $2s=a+b+c$, as in Prob. 6.

$$\mathbf{\cos A=\frac{2s(s-a)}{bc}-1.}$$

Similarly, $\quad \mathbf{\cos B=\dfrac{2s(s-b)}{ca}-1, \quad \cos C=\dfrac{2s(s-c)}{ab}-1.}$

Ex. 73. ABCD *is a parallelogram in which*

$$AB = 8{\cdot}4 \textit{ cm.}, \; BC = 7{\cdot}8 \textit{ cm.}, \textit{ and } BD = 9{\cdot}4 \textit{ cm.}$$

Find the angle BAD *and the length of the diagonal* CA.

The parallelogram is shewn in Fig. 60. Since AD = BC, we have for the triangle ABD, $2s = 8{\cdot}4 + 7{\cdot}8 + 9{\cdot}4 = 25{\cdot}6$ cm.;

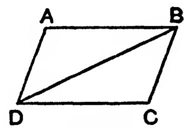


FIG. 60.

$$\therefore \; s - BD = 12{\cdot}8 - 9{\cdot}4 = 3{\cdot}4 \text{ cm.};$$

$$\therefore \; \cos A = (25{\cdot}6 \times 3{\cdot}4)/(7{\cdot}8 \times 8{\cdot}4) - 1$$
$$= 1{\cdot}3284 - 1 = 0{\cdot}3284.$$

Hence from the tables, $A = 70° \, 50'$.

To find the diagonal CA, we must use the first form of the cosine rule.

$$\therefore \; CA^2 = AB^2 + BC^2 - 2AB \,.\, BC \,.\, \cos ABC$$
$$= 8{\cdot}4^2 + 7{\cdot}8^2 - 2 \times 8{\cdot}4 \times 7{\cdot}8 \cos (180° - A)$$
$$= 70{\cdot}56 + 60{\cdot}84 + 15{\cdot}6 \times 8{\cdot}4 \cos A = 171{\cdot}44;$$
$$\therefore \; CA = \sqrt{171{\cdot}44} = 13{\cdot}1 \text{ cm.}$$

EXERCISES 11B.

Solve each of the following triangles by the cosine rule:

1. $a = 28$ in., $b = 53$ in., $c = 45$ in. **2.** $a = 58$ ft., $b = 51$ ft., $c = 41$ ft.

3. $a = b = 65$ cm., $c = 32$ cm. **4.** $a = 97$ m., $b = 72$ m., $c = 65$ m.

5. $a = 35$ yd., $b = 73$ yd., $c = 52$ yd.

6. $a = 82$ ch., $b = 57$ ch., $c = 89$ ch.

7. $a = 21$, $b = 61$, $c = 68$. **8.** $a = b = 250$, $c = 154{\cdot}5$.

9. Find the angles and the other diagonal in a parallelogram ABCD in which AB = 97 yd., AD = 78 yd., and BD = 95 yd.

10. Calculate the angles and the diagonal BD of a parallelogram ABCD in which AB = 41 m., BC = 58, m. and CA = 51 m.

76. Case 2. When Two Sides and the Included Angle are given. Here the triangle may be solved either by dividing it into two right-angled triangles, or by using the cosine rule.

Ex. 74. *Solve the triangle in which*

$$c = 8{\cdot}6 \text{ ft.}, \quad a = 6{\cdot}7 \text{ ft.}, \quad \textit{and} \quad B = 52° \, 24'.$$

(i) Draw CF perpendicular to AB (see Fig. 59, p. 123).

Then
$$FC = a \sin B = 6{\cdot}7 \times 0{\cdot}7923 = 5{\cdot}308 \text{ ft.},$$
$$FB = a \cos B = 6{\cdot}7 \times 0{\cdot}6101 = 4{\cdot}088 \text{ ft.},$$
$$\therefore \; AF = 8{\cdot}6 - 4{\cdot}088 = 4{\cdot}512 \text{ ft.},$$

and
$$\tan A = FC/AF = 5{\cdot}308/4{\cdot}512 = 1{\cdot}176 = \tan 49° \, 38' ;$$
$$\therefore \; A = \mathbf{49° \, 38'}.$$

Hence, $C = 180° - (A + B) = 180° - 102° \, 2' = \mathbf{77° \, 58'}.$

Finally, $b = CA = AF/\cos A = 4{\cdot}512/0{\cdot}6477 = \mathbf{6{\cdot}97}$ ft.

If a table of secants were available, we could say that
$$b = AF \sec A = 4{\cdot}512 \times 1{\cdot}5439 = 6{\cdot}97 \text{ ft.}$$

(ii) No perpendicular need be drawn if we make a direct application of the cosine rule ; thus
$$b^2 = c^2 + a^2 - 2ca \cos B = 73{\cdot}96 + 44{\cdot}89 - 17{\cdot}2 \times 6{\cdot}7 \times 0{\cdot}6101$$
$$= 118{\cdot}85 - 70{\cdot}31 = 48{\cdot}54 ;$$
$$\therefore \; b = \sqrt{48{\cdot}54} = \mathbf{6{\cdot}97}.$$

To find A, we have $2s = 6{\cdot}7 + 6{\cdot}97 + 8{\cdot}6 = 22{\cdot}27$ ft.

and
$$s - a = 11{\cdot}135 - 6{\cdot}7 = 4{\cdot}435 \text{ ft.},$$
$$\therefore \; \cos A = (22{\cdot}27 \times 4{\cdot}435)/(6{\cdot}97 \times 8{\cdot}6) - 1$$
$$= 1{\cdot}6477 - 1 = 0{\cdot}6477 ;$$
$$\therefore \; A = \mathbf{49° \, 38'}$$

and
$$C = 180° - (A + B) = 180° - 102° \, 2' = \mathbf{77° \, 58'}.$$

EXERCISES 11c.

Solve each of the following triangles :

1. $a = b = 7{\cdot}8$ in., $C = 60°$. **2.** $a = 15{\cdot}7$ ft., $b = 13{\cdot}2$ ft., $C = 32° \, 48'$.

3. $a = 24$ cm., $b = 37$ cm., $C = 71° \, 4'$.

4. $b = 18{\cdot}5$ cm., $c = 11{\cdot}4$ cm., $A = 72° \, 3'$.

5. $b = 9{\cdot}5$ yd., $c = 19{\cdot}3$ yd., $A = 60° \, 31'$. **6.** $b = 2$, $c = \sqrt{6}$, $A = 75°$.

7. $c = 52$ m., $a = 41$ m., $B = 12° \, 38'$.

8. $b = 17{\cdot}37$ ch., $c = 21{\cdot}07$ ch., $A = 47°$.

9. $a = 398{\cdot}9$, $c = 496{\cdot}5$, $B = 72° \, 18'$. **10.** $b = 215$, $c = 197$, $A = 73°$.

77. Case 3. When One Side and Two Angles are given. In this case, as in the previous one, the triangle may be solved by dividing it into two right-angled triangles.

Ex. 75. *Solve the triangle in which* $C = 53° 12'$, $A = 67° 36'$ *and* $b = 18{\cdot}4$ m.

Again referring to Fig. 59 (p. 123), note that the perpendicular CF is drawn to one of the unknown sides. This should always be done.

Now $\quad B = 180° - (C + A) = 180° - 120° 48' = 59° 12'$,

and $\quad AF = b \cos A = 18{\cdot}4 \times 0{\cdot}3811 = 7{\cdot}012$,

$FC = b \sin A = 18{\cdot}4 \times 0{\cdot}9245 = 17{\cdot}01$,

$FB = FC \cot B = 17{\cdot}01 \times 0{\cdot}5961 = 10{\cdot}138$;

$\therefore\ c = AF + FB = 7{\cdot}012 + 10{\cdot}138 = \mathbf{17{\cdot}15}$ m.

Finally, $a = FC/\sin B = FC \operatorname{cosec} B = 17{\cdot}01 \times 1{\cdot}1642 = \mathbf{19{\cdot}8}$ m.

78. The Sine Rule. We can solve a triangle in this case without first finding the segments into which the perpendicular divides the base, by means of a very simple and useful rule connecting the sines of the angles with the lengths of the sides.

PROBLEM 21. *In any triangle, to find a relation between the sines of the angles and the sides opposite those angles.*

Let ABC (Fig. 61) be any triangle; draw the perpendiculars AD, BE from A to BC and from B to CA respectively.

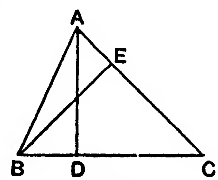


FIG. 61.

Then from the triangle ABD, $AD = c \sin B$,

and " " ACD, $AD = b \sin C$;

$\therefore\ c \sin B = b \sin C$, or $\sin B/b = \sin C/c$.

Also from the triangle BAE, $BE = c \sin A$,

and " " BCE, $BE = a \sin C$;

$\therefore\ c \sin A = a \sin C$, or $\sin A/a = \sin C/c$.

Hence, since $\sin B/b$ and $\sin A/a$ are each equal to $\sin C/c$, we have

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

This important relation is called the **Sine Rule** of a triangle.

Ex. 76. *Solve the triangle in which* $b=562$ *yd.*, $B=46^\circ\ 12'$ *and* $C=82^\circ\ 36'$.

Here we have $A=180^\circ-(B+C)=180^\circ-128^\circ\ 48'=51^\circ\ 12'$.

Now, from the sine rule,

$$\frac{\sin A}{a}=\frac{\sin B}{b} \quad \text{and} \quad \frac{\sin C}{c}=\frac{\sin B}{b},$$

$$\therefore \quad a=\frac{b\sin A}{\sin B}=\frac{562\times 0{\cdot}7793}{0{\cdot}7218}=606{\cdot}7 \text{ yd.}$$

$$c=\frac{b\sin C}{\sin B}=\frac{562\times 0{\cdot}9917}{0{\cdot}7218}=772{\cdot}1 \text{ yd.}$$

This rule enables us to apply logarithms directly to calculate the unknown sides.

EXERCISES 11D.

Solve each of the following triangles:

1. $A=63^\circ\ 12'$, $B=58^\circ\ 24'$, $c=125$ ft.
2. $A=46^\circ\ 24'$, $C=43^\circ\ 36'$, $b=29$ yd.
3. $A=47^\circ\ 33'$, $B=84^\circ\ 54'$, $a=15{\cdot}2$ m.
4. $B=52^\circ$, $C=63^\circ$, $a=973{\cdot}6$ yd.
5. $A=100^\circ$, $B=42^\circ\ 46'$, $c=108$ m.
6. $A=68^\circ\ 54'$, $B=21^\circ\ 6'$, $b=36$ in.
7. $A=56^\circ\ 18'$, $C=67^\circ\ 48'$, $b=327$ cm.
8. $A=111^\circ\ 6'$, $B=48^\circ$, $b=25$ chains.
9. $A=88^\circ\ 36'$, $B=31^\circ\ 54'$, $c=10{\cdot}6$.
10. $B=75^\circ\ 24'$, $C=62^\circ\ 42'$, $a=324{\cdot}1$.

79. Case 4. When Two Sides and the Angle opposite one of them are given. Before proceeding to solve a triangle in this case, we shall first consider how it may be constructed.

PROBLEM 22. Construct a triangle ABC, *having given the sides* c, a *and the angle* A. *Examine carefully all the possible cases.*

Draw the angle $XAB=A$ (Fig. 62). Cut off $AB=c$, and with centre B and radius equal to a describe a circle. Suppose this circle cuts AX in two points C, C′ on the same side of A as X, then clearly both the triangles ABC, ABC′ satisfy the given conditions,

and it should be observed that $C' = \angle C'CB = 180° - C$, *i.e.* C, C′ are supplementary. Also $BC < AB$.

The condition for the circle to cut AX must be examined. Draw BE perpendicular to AX, then $BE = AB \sin A = c \sin A$, and the circle will cut AX if $BC > BE$, *i.e.* if $a > c \sin A$.

If $a > c$, *i.e.* $BC > AB$, C will lie to the left of A; then only the triangle ABC′ will satisfy the given conditions.

Further, if $BC = BE = BC'$, *i.e.* $a = c \sin A$, then C, C′, E coincide, and one right-angled triangle ABC results. This will fulfil the given conditions if A is acute, *i.e.* $a < c$, as in Fig. 63.

Finally, if $a < c \sin A$, then the circle does not reach AX, as in Fig. 64, and no triangle can be constructed.

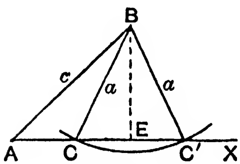


FIG. 62.

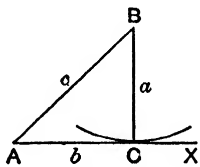


FIG. 63.

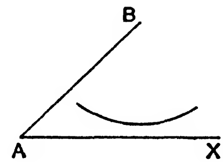


FIG. 64.

Hence, summing up the results, **when two sides c, a, and a non-included angle A of a triangle are given,**

(i) **there are two solutions if $a > c \sin A$ and $a < c$, in which case the respective angles opposite side c are supplementary.**

(ii) **there is one solution if $a > c \sin A$ and $a > c$.**

(iii) **there is one solution if $a = c \sin A$ and $a < c$, in which case $c^2 = a^2 + b^2$.**

(iv) **there is no solution if $a = c \sin A$ and $a > c$.**

(v) **there is no solution if $a < c \sin A$.**

Since there are thus several possible solutions, as well as impossible ones according to the nature of the data, this case of solution is called the **ambiguous case.**

Ex. 77. *If $A = 30° 30'$, $b = 48$, solve the triangle if possible when a is* (α) 25·42, (β) 24·36 *and* (γ) 23·84.

The perpendicular from C to $AB = b \sin A = 48 \times 0{\cdot}5075 = 24{\cdot}36$.

Hence (α) when $a = 25{\cdot}42$ there will be two solutions by (i).

By the sine rule, $\sin B = b \sin A/a = 24{\cdot}36/25{\cdot}42 = 0{\cdot}9585$;

$\therefore$ From the tables, $B = 73° 26'$ and $\sin(180° - B) = \sin B$, so that $180° - B$ is the corresponding angle of the second triangle.

This verifies the fact deduced in Prob. 22 that the two angles opposite b are supplementary,

$$\therefore\ B = 73^\circ\ 26' \quad \text{or} \quad 106^\circ\ 34'.$$

Hence $C = 180^\circ - (A + B) = 76^\circ\ 4'$ or $42^\circ\ 56'$.

To find c, we have, by the sine rule, $c = b \sin C/\sin B$, which, on putting in the two pairs of corresponding values of B and C, gives

$$c = 48{\cdot}6 \quad \text{or} \quad 34{\cdot}11.$$

(β) With $a = 24{\cdot}36$, $\sin B = 24{\cdot}36/24{\cdot}36 = 1$;

$$\therefore\ B = 90^\circ,$$

and there is only one triangle which is right-angled.

Thus C and A are complementary, and $C = 59^\circ\ 30'$.

Also $c = b \sin C = 48 \times 0{\cdot}8616 = 41{\cdot}36$.

(γ) With $a = 23{\cdot}84$, $\sin B = 24{\cdot}36/23{\cdot}84 = 1{\cdot}022$, which is impossible, since the sine cannot exceed unity,

$\therefore$ **there is no solution.**

EXERCISES 11E.

Find the angle opposite the greater of the given sides, and the third side in each of the following triangles:

1. $B = 30^\circ$, $b = 52$ in., $c = 78$ in.
2. $A = 69^\circ\ 24'$, $b = 12{\cdot}5$ ft, $a = 11{\cdot}7$ ft.
3. $B = 54^\circ$, $b = 90{\cdot}2$ cm., $c = 115$ cm.
4. $C = 56^\circ\ 6'$, $a = 17$ yd., $c = 16{\cdot}6$ yd.
5. $B = 65^\circ\ 30'$, $a = 54$ m., $b = 65$ m.
6. $B = 68^\circ\ 54'$, $c = 27$ ch., $b = 31{\cdot}1$ ch.
7. $B = 16^\circ\ 16'$, $b = 7$, $c = 25$.
8. $C = 63^\circ$, $b = 216$, $c = 183$.
9. $A = 40^\circ\ 24'$, $a = 35$, $b = 54$.
10. $A = 60^\circ$, $a = b = 38{\cdot}5$.

EXERCISES 11F. (*Miscellaneous.*)

1. Solve the triangle in which $a = 562{\cdot}3$ ft., $b = 349{\cdot}5$ ft., $c = 456{\cdot}8$ ft.

2. Find the remaining angles of a triangle in which $a = 7235$ links, $b = 4635$ links, and $C = 78^\circ\ 26'$.

3. Calculate a and C in a triangle where $A = 55^\circ$, $B = 65^\circ$ and $c = 270$ yd.

4. Find A and B when in a triangle $a = 2b$ and $C = 75^\circ$.

5. Determine the angles in a triangle in which $a = 187$ cm., $b = 165$ cm., and $c = 88$ cm.

6. Calculate the values of a and C when in a triangle $b=486$ m., $c=643$ m. and $B=48° 18'$.

7. Find the longest side of a triangle in which $B=65° 36'$, $C=78°$ and $a=424$ ft.

8. ABCD is a parallelogram in which AB = 12 cm., BC = 5 cm. and $\angle$ BAD $=68° 54'$. Find the lengths of the diagonals BD and CA.

9. Find the angles B and C of a triangle in which $b=535$ ft., $c=465$ ft. and $A=51° 20'$.

10. If in a triangle $a=7$, $b=10$, $c=13$, find its smallest angle.

11. Find the values of the angle C of a triangle in which $b=762$ ft., $c=735$ ft. and $B=51° 42'$.

12. The difference between the base and the perpendicular of a right-angled triangle is one-fifth of their sum. Find the angles of the triangle.

13. If two sides of a triangle are 56 and 87 yards long and the area of the triangle is 984 sq. yd., find the possible values of the angle between the given sides. (J.M.B.)

14. Solve a triangle in which $a=12{\cdot}35$ in., $B=35° 30'$, $C=64° 24'$. (J.M.B.)

15. In the triangle ABC, AB = 17 inches, AC = 19 inches, the angle ABC $=31° 30'$. What is the value of the area? (J.M.B.)

16. If in a triangle ABC, $2a=3b$, and $A=2B$, express c in terms of b and hence find the angles of the triangle.

17. The sides of a triangle are in the ratios $4:10:11$; find the greatest angle of the triangle. (J.M.B.)

18. Two angles of a triangle are 43° and 65°, and the area of the triangle is 1000 sq. yd. Calculate the length of the longest side of the triangle. (C.P.)

19. The diagonals of a parallelogram are 35 and 45 inches long, and inclined to one another at 80°. Calculate the sides and angles of the parallelogram. (D.U.)

20. Three angles A, B, C of a quadrilateral are each 75°; AB = 10 inches and AD = 5 inches. Find the lengths of BC and CD. (L.S.)

21. A ship C is seen by an observer at A to be in a direction 32° 7′ N. of W.; another observer B, 800 yards to the west of A, sees the ship at the same time to be 43° 49′ N. of W. How far is the ship from A? (L.S.)

22. A triangle has base 4 in., height 3·5 in. and one of its base angles is 120°. Find the side opposite to this angle to the nearest hundredth of an inch. (L.M.)

23. Prove the formula $a^2 = b^2 + c^2 - 2bc \cos A$.

If A, B, C are three points in order on a straight line, and P any point not on the line, prove that

$$AB \cdot CP^2 + BC \cdot BP^2 - AC \cdot BP^2 = AB \cdot BC \cdot AC. \qquad \text{(L.M.)}$$

24. Explain what is known as the "Ambiguous Case" in the solution of triangles, stating the circumstances in which the ambiguity arises.

In a triangle ABC, $a=75$, $b=92$, $A=48°$. Find the other angles of the triangle. (L.M.)

25. In a triangle ABC, $a=3$, $b=26$, $c=25$; BC is produced to D so that CD $=22$. Find AD and angles BAC, CAD.

26. In a triangle ABC, $a=28$, $b=17$ and $\cos B=0{\cdot}8$; use the cosine rule to find the two values of c, and give the area of the triangle in each case.

27. ABC is a triangle whose bisector of A meets BC in D. If $b=2$, $c=5$ and AD $=2{\cdot}5$, find the angle A and the side a.

28. If a, b are the lengths of the sides of a parallelogram, and θ is the acute angle between them, shew that the diagonals are of length $\sqrt{a^2+b^2 \pm 2ab\cos\theta}$.

Find the diagonals of a rhombus of side 8 inches and angle 60°.

29. A town B is 153 miles from a town A in a direction 10° N. of W.; a town C is 254 miles from A in a direction 35° N. of E. Find the distance and direction of C from B. (L.M.)

30. Devizes is $9\frac{1}{2}$ miles from Chippenham in a direction 37° E. of S., Melksham is 7 miles from Chippenham in a direction 11° W. of S. Calculate how many miles Devizes is from Melksham, and in what direction. (J.M.B.)

31. In the triangle ABC, the angle BAC $=43°\ 30'$, the angle ACB $=47°\ 45'$. The line BD divides the angle ABC into two parts, one of which is double the other, and meets AC in D. What is the ratio of AD to DC, AD being the shorter segment? (J.M.B.)

32. In a triangle ABC, the bisector of the angle A meets BC in D; calculate, by the sine rule, the length of AD when $b=24$, $c=28$, and $A=49°$.

33. Compare the areas of the two triangles in which $A=31°\ 47'$, $a=85$, and $c=160$.

34. ABCD is a quadrilateral in which AB $=16$, CD $=25$, DA $=15$, $\angle$ BAD $=66°\ 25'$ and $\angle$ BCD $=36°\ 52'$. Find the length of BC.

35. In a triangle ABC, $c=153$, $\tan A=1$, and $\tan B=2$; find, without using tables, the remaining sides and the value of $\tan C$.

CHAPTER XII

APPLICATIONS OF THE SOLUTION OF TRIANGLES

80. Application of the Right-angled Triangle. In Chapter II. we have already discussed some simple practical problems on the solution of right-angled triangles. Here a more difficult case will be considered.

Ex. 78. *To find the width of a river, two objects* A, B *on the left bank are observed from two positions* C, D *on the right bank, which are in the line perpendicular to* AB *through* B. *At* C, *which is* 35 *ft. from the right bank,* $\angle ACB = 22°$ *; at* D, $\angle ADB = 20°$, *and* $CD = 10$ *ft. Calculate the width of the river and the distance between* A *and* B.

Let BD (Fig. 65) cut the right bank in E, then

$EC = 35$ ft., $CD = 10$ ft., so that $ED = 45$ ft.

Let the width BE, of the river, be x ft., then

$$AB = BD \tan 20° = (x + 45) \times 0{\cdot}364,$$

and $$AB = BC \tan 22° = (x + 35) \times 0{\cdot}404,$$

$$\therefore\ (x + 45) \times 0{\cdot}364 = (x + 35) \times 0{\cdot}404,$$

from which $$x = \mathbf{56}\ \text{ft.}$$

To find AB, we have

$$AB = (x + 45) \times 0{\cdot}364 = 101 \times 0{\cdot}364 = 36{\cdot}76\ \text{ft.},$$

or $$= \mathbf{36{\cdot}8}\ \text{ft. nearly.}$$

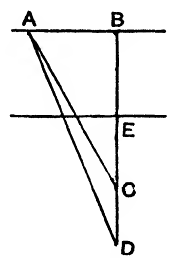


FIG. 65.

81. Application of the General Triangle. In addition to the area problems in surveying in which a district is divided up into triangles, there are many practical cases where an application of the solution of a general triangle is useful. The following example will illustrate this.

Ex. 79. *A straight road* AC, *inclined at an angle of* 17° *to the horizontal, leads from the sea-level* A *to a vertical monument* CD, *the top of which is observed from two points* A, B *on the road. At* A *the angle* CAD $=24°$, *and at* B, *the angle* CBD $=37°$. AB $=90$ *ft.*; *calculate the height of* D *above the sea-level, and the height* CD *of the monument.*

From the diagram (Fig. 66), it is evident that we have to calculate the distances FD and CD.

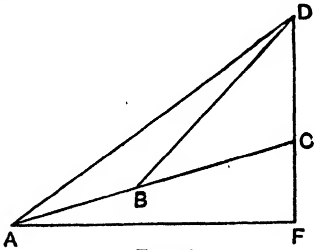


FIG. 66.

Now FD = AD sin DAF, and from the sine rule,

$$AD/\sin ABD = AB/\sin ADB,$$

so that $$AD = AB \sin ABD/\sin ADB;$$

and $$FD = AB \sin ABD \;.\; \sin DAF/\sin ADB.$$

But $\sin ABD = \sin DBC = \sin 37° = 0{\cdot}6018$.

$$\angle FAD = \angle DAC + \angle CAF = 24° + 17° = 41°,$$

and $$\angle ADB = \angle CBD - \angle CAD = 37° - 24° = 13°;$$

$$\therefore\; FD = 90 \times 0{\cdot}6018 \times \sin 41°/\sin 13°$$
$$= 90 \times 0{\cdot}6018 \times 0{\cdot}6561/0{\cdot}225 = \mathbf{158} \text{ ft.}$$

To find CD, we have

$$AF = FD \cot FAD = FD \cot 41° = 158 \times 1{\cdot}1504,$$

and $$CF = AF \tan FAC = AF \tan 17°$$
$$= 158 \times 1{\cdot}1504 \times 0{\cdot}3057 = 55{\cdot}6 \text{ ft.};$$

$$\therefore\; CD = 158 - 55{\cdot}6 = \mathbf{102{\cdot}4} \text{ ft.}$$

82. Application to Three Dimensions. It is sometimes venient to take observations in the same vertical plane, and as a consequence, some problems can only be solved from measurements taken in two or more vertical planes; hence three dimensions are necessary. The following is a typical example.

Ex. 80. *A mountain peak is observed from two stations* P, Q *situated* 679 *ft. apart on a horizontal stretch of ground at the sea-level. At* P, *which is due west of the peak, the angle of elevation is* 65° 32′, *whilst at* Q, *which is due south of the peak, the angle of elevation is* 63° 15′. *Calculate the height of the peak above the sea-level.*

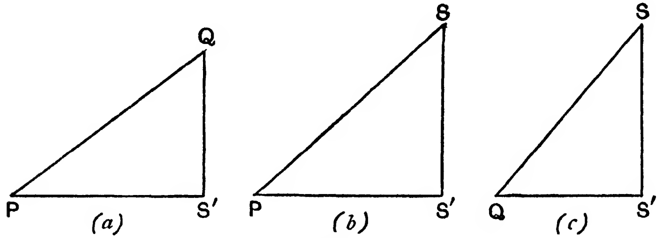


FIG. 67.

Let P, Q be the stations and S′ (Fig. 67*a*) the foot of the peak on the horizontal plane of the stations; then $\angle$PS′Q is a right angle, and PQ = 679 ft.

Taking a vertical plane through P and the peak, we have the right-angled triangle PS′S (Fig. 67*b*), where S is the summit, and $\angle$S′PS = 65° 32′.

If, therefore, h = height S′S of the peak in feet,

$$PS' = h \cot 65^\circ\, 32' = h \times 0{\cdot}455.$$

Again, taking a vertical plane through Q and the peak S, we have a right-angled triangle QS′S (Fig. 67*c*), in which S′S = h ft., and $\angle$S′QS = 63° 15′;

$$\therefore\ QS' = h \cot 63^\circ\, 15' = h \times 0{\cdot}504.$$

Hence from Fig. 67*a*, $PQ^2 = PS'^2 + S'Q^2$,

i.e.
$$679^2 = h^2(0{\cdot}455^2 + 0{\cdot}504^2)$$
$$= h^2 \times 0{\cdot}679^2;$$
$$\therefore\ h = 1000 \text{ ft.}$$

EXERCISES 12.

1. A tower stands on a level plain. A person A standing on the plain 50 feet west of the tower observes the angle of elevation of its top to be 34°, while a second person B standing due east of the tower observes the angle of elevation to be 27°. Find the height of the tower and the distance of B from its foot. (U.L.C.I.)

2. A person on a ship which is sailing on a straight course, observes two lighthouses, one of which, L, is due north of the other, M. When M is due east, L bears 45° to the east of north, and when L is due east, M bears 60° to the east of south. Find the direction in which the ship is sailing.

3. P and Q are two stations 1000 yards apart. The line PQ bears east and west. At P a rock bears 42° west of south; at Q it bears 35° east of south. Calculate the distance of the rock from the shore.

4. From a ship steaming at 12 miles per hour in a direction 71° 36′ E. of N., the bearing of a lighthouse is observed to be 14° 12′ E. of N. at 8 p.m., and to be 33° W. of N. at 9.30 p.m. Find the distance of the lighthouse from the ship's path, neglecting the curvature of the earth.

5. A train travelling at a uniform rate of 21 miles per hour takes 3·75 minutes to go from a station P along a straight line to the next station Q. From a point A in the same horizontal plane, it is observed that

$$\angle APQ = 3\angle PQA = \tfrac{1}{2}\angle PAQ.$$

Find the distances in yards from A to P and to Q respectively.

6. The angle of elevation of the top of a house from a certain point on the ground is observed to be 47° 24′. On receding 25 ft. further from the house, the angle of elevation of the top is now 32° 47′. Calculate the height of the house. (N.U.T.)

7. By observations on a lighthouse L on an island, from two stations P and Q, 1250 yards apart, on the shore of the mainland, the angle LPQ is found to be 62°, and the angle LQP 51°. If the shore between P and Q is straight, what is the shortest distance to the lighthouse from the shore? (N.U.T.)

8. Two ships B and C are seen from a lighthouse A; B is 7·94 miles distant in the direction 29° 13′ S. of W., and C 9·38 miles distant in the direction 10° 41′ W. of S. What is the direction of C as seen from B? (L.M.)

9. In a survey it was necessary to continue a line PQ past an obstacle which cannot be seen over. A line QB of length 140 yards was measured off at right angles to PQ, and from the point B, lines BD and BE are drawn which clear the obstacle; the angles QBD and QBE are 32° and 45° respectively. Find the lengths of BD and BE so that DE and PQ may be in the same straight line. (U.L.C.I.)

10. From the top of a vertical cliff 100 ft. high, forming one bank of a river, the angles of depression of the top and bottom of the vertical cliff forming the opposite bank are 28° 40′ and 64° 30′ respectively. Find the height of the cliff and the width of the river.

11. To get from a place A to a place C, it is necessary to go along two straight roads AB, BC, inclined to one another at an angle of 85°. Calculate what distance would be saved if it were possible to go along the straight road AC, it being given that AB$=2\frac{1}{2}$ miles and BC$=3\frac{3}{8}$ miles. (U.L.C.I.)

12. It is found by walking towards a chimney a distance of 75 feet along a straight line through the base that the angle of elevation of its top is changed from 42° to 59°. Find the height of the chimney. (N.U.T.)

13. Two chimneys AB and CD are of equal height. A person standing between them in the line AC joining their bases observes the elevation of AB to be 60°. After walking 100 ft. in a direction at right angles to AC he finds the elevation of AB to be 45° and of CD 30°. Find the height of the chimneys and the distance between them. (J.M.B.)

14. From their point of intersection, Danson Lane runs in a direction of 18° east of North ; Holderness Road runs in a direction of 22° north of East. Kent Street, which runs from the latter road to the former, has a direction 56° west of North, and is 410 yards long. How far is it along Holderness Road from Danson Lane to Kent Street ? (J.M.B.)

15. At a point on a level plain the elevation of the top of a mountain is 22°, and at another point on the plain, a mile further away in a direct line, the elevation is 9° 48′. Find the height of the mountain above the plain. (J.M.B.)

16. A lighthouse L is observed from a station A on shore. B is another station on the shore 1065 links from A, and the angle LAB is found to be 85° 36′. At B, a pier BC, 585·9 links long is at right angles to AB, and on observing L from C the angle LCB is found to be 126° 30′. Calculate the shortest distance of the lighthouse from the shore AB.

17. From a point on a river bridge, the angle of depression of the foot of a telegraph pole, 32 ft. high, on the bank is 47° 32′, while the angle of elevation of the top of the pole from the same point is 19° 16′. What is the height of the observation point above the bank ? (N.U.T.)

18. A and B are two points 500 ft. apart on the straight towing path of a canal. C is the site of a house on the opposite bank 100 yards from the canal. The angles ABC and BAC are found to be 42° 23′ and 65° 13′ respectively. Find the width of the canal. (N.U.T.)

19. The side of a hill is a plane surface and slopes at 27° to the horizontal. Two men walk from the same point at the foot of the hill. One man walks 510 yards in the direction of the steepest slope, the other 380 yards in a direction making 43° with that of the first man. Find the difference in level between the two men in their new positions and the actual distance between them. (U.L.C.I.)

20. A man observes that the angle of elevation of an aeroplane at a certain instant is 33° and that 54 seconds later it is 43°. Assuming that the aeroplane is flying at the rate of 80 miles an hour in a horizontal line directly over the observer, find its height above the ground, and the time which elapses after the first observation before it is overhead. (L.M.)

21. A vertical tower PQ is observed from two stations A, B in the same horizontal plane as the foot Q of the tower. A is due south and the observed angle of elevation of P is 31° 48′. B is 336 ft. due east of A, and the elevation of P there is 17° 12′. Find the height of the tower.

22. When a traveller is at a place A, the top of a mountain is seen to the north-east at an elevation of 21° ; when he arrives at B, in the same horizontal plane as A, the mountain is due east and the elevation 15°. The top of the mountain is known to be 7300 ft. above the level of A and B. Calculate (i) the distances of A and B from the projection of the top of the mountain on the horizontal plane through A and B, (ii) the distance from A to B. (L.M.)

23. A tower stands on a horizontal plane at a distance of 500 yd. from the foot of a mountain; an observer, whose distance up the mountain slope is 1000 yd., looking over the top of the tower can just see the nearer bank of a river whose distance behind the tower is 100 yd. When he has walked half-way down the slope, which is to be taken of a uniform gradient, he can just see the further bank of the river when he looks over the tower. Given that the river is 50 yd. wide, find the height of the tower and the inclination of the mountain side to the horizontal. (J.M.B.)

24. A, B, P, Q are four points in the same horizontal plane, and B is 650 yards due East of A. From A, P bears N. 19° E., and Q bears N. 50° E. From B, P bears N. 46° W., and Q bears N. 25° W. Calculate the distances AP, AQ, PQ. (L.M.)

25. Walking along a straight level road in a direction N.W., I notice two spires P, Q in a straight line with me on a bearing N. 20° E., P being the nearer spire. After walking 4 miles further along the road, P bears E. 22° S., and Q bears E. 26° N. Find the distance between the spires. (L.M.)

26. A line AB, 100 yards in length, is measured along a straight road inclined to the horizontal at an angle of 5°. At the lower extremity A the angle of elevation of an object P, beyond B, in the same vertical plane as AB is 32° ; at the upper extremity B, the angle of elevation of P is 38°. Find the height of the point P above the horizontal plane through A. (J.M.B.)

27. A sailing boat has to go from a certain point to another 21 miles off in a direction S. 40° E. After sailing 10 miles S. 35° E. it alters its direction 15° more to the east and sails another 10 miles. How far will it then be from its destination, assuming there is no current ? (L.M.)

28. A steamer is going down Channel at constant speed in a straight line when a passenger P sees two landmarks A, B which are known to be 6510 yards apart, and observes that PA, PB make angles 85° 15′ and 42° 20′ with the steamer's course. Forty-two minutes later he notices that the steamer is crossing the line AB produced at an angle 31° 35′. Find the speed of the steamer in knots. A knot is 6080 feet per hour. (L.M.)

29. ABC is a triangle in which A = 63°, B = 47° and AB = 58 ft. AB is produced to P so that the angle BCP = 25°. Calculate the length of BP to the nearest foot.

30. A and B are two points in a horizontal plane, 153 feet apart, and C is a point in the same plane such that the angles CAB and CBA contain 73° and 39° 15′ respectively. Find the length of the perpendicular distance of C from AB, and the approximate size of the angle between that perpendicular and the straight line joining C to the middle point of AB. (C.P.)

31. Two boats at sea are observed from a point on a cliff 350 feet high. The bearing and angle of depression of one are N. 52° 20′ E. and 20° 35′, and of the other N. 22° 50′ W. and 13° 50′. Find how far each boat is north of the observer, how far east and west of him respectively, and how far the second boat is north and west of the first. (L.M.)

32. Two chief landmarks on a stretch of flat country are a lake distant 3 miles from an aerodrome, and a wood distant 5 miles from the lake. An observer in an airship vertically above the aerodrome notes that the bearing of the lake is S. 52° W., and its angle of depression is 20°. Ten minutes later when vertically above the wood he notes that the bearing of the lake is S. 78° E., and its angle of depression is 22°. Find the average rate in miles per hour at which the airship has been moving horizontally, and the average rate in feet per minute at which it has been rising. (L.M.)

33. A lighthouse is observed from two stations A, B on a horizontal cliff in a straight line passing through the lighthouse. AB = 48·6 feet, and the angles of elevation and depression of the top and bottom of the lighthouse at A are 20° 54′ and 19° 42′ respectively, whilst at B the elevation is 16° 48′. Find the height of the lighthouse.

34. An aeroplane flying at 52·5 miles an hour in a horizontal line due south is observed by two men A and B. When the aeroplane is due east of A he observes that its angle of elevation is 45°. Two seconds later, B, who is due north of the aeroplane, observes that its angle of elevation is 68° 12′. The distance between A and B is 902 ft.; calculate the height of the aeroplane in feet.

CHAPTER XIII

MEASUREMENT OF VOLUME

83. Volume. If we take a flat sheet of paper so thin that its thickness may be neglected in comparison with its size, the sheet obviously represents an area because it possesses two dimensions only, viz. length and breadth. If, however, a large number of similarly shaped sheets be placed face to face, like the leaves of this book when closed, the thickness can no longer be neglected; consequently we have not only length and breadth, but also the thickness to take into consideration. There are now three dimensions, and a body, capable of retaining its shape, like a block of wood, and possessing three dimensions is called a **solid**. The amount of space occupied by such a body is called its **volume or cubical contents.**

84. Measurement of Volume. The dimensions of a solid body are measured in three mutually perpendicular directions; hence the unit of volume chosen is a solid whose length, breadth and thickness are not only perpendicular to each other, but are also equal. Such a solid is called a **unit cube**, and one such unit, representing a cubic inch, is shewn on the right in Fig. 68. It will be seen from the figure that a cube contains 12 edges forming the sides of 6 square faces.

The length of the edge chosen may be any one of the linear units of either the British or Metric systems, according to the size of the body whose volume is required. If the edge be one inch in length, the volume of the unit cube is called a **cubic inch**; if the edge be one centimetre, the volume of the cube is known as **a cubic centimetre**, and so on. The process of finding the volume of a solid consists, therefore, in finding the number of unit cubes contained in that solid.

85. British Measures of Volume. A cubic inch is shewn upon a small scale on the right hand of Fig. 68. If we place twelve of

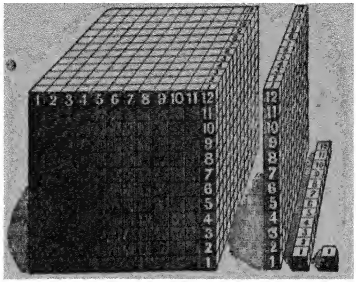


FIG. 68.—A cubic foot built up of cubic inches.

these in a row, we have a stick one foot in length, and one square inch in section. Twelve of these sticks laid one upon another build up a slab containing $12 \times 12 = 144$ cubic inches, and if twelve of these slabs be placed together as shewn, we have a cube each of whose edges is 1 foot in length, *i.e.* the volume of the cube is one cubic foot.

Hence the number of cubic inches in this cubic foot is $144 \times 12 = 1728$, *i.e.* 12^3.

Exactly the same reasoning will shew that a cubic yard contains 3^3 or 27 cubic feet. Hence the following tables of cubic measure.

British Measure of Volume.

12^3 or 1728 cubic inches = 1 cubic foot
3^3 or 27 cubic feet = 1 cubic yard

For measuring the internal volumes, or capacities of hollow bodies, used for holding liquids, the following table is used.

2 pints = 1 quart
4 quarts = 1 gallon.

In this connection it is only the gallon that is generally used, and an Imperial gallon is defined as the quantity of distilled water which weighs 10 lb. under standard conditions. It is equal to 277·274 cubic inches, and an approximation useful in most cases is 277·25 cub. in.

Ex. 81. *The capacity of a cistern is* 221·8 *cub. ft.; how many gallons will it hold taking one gallon to be equivalent to* 277·25 *cubic inches?*

$$\begin{aligned}\text{Capacity} &= 221{\cdot}8 \text{ cub. ft.}\\ &= 221{\cdot}8 \times 1728 \text{ cub. in.}\\ &= (221{\cdot}8 \times 1728)/277{\cdot}25 \text{ gallons}\\ &= \frac{2218 \times 1728 \times 4}{10 \times 1109} = \mathbf{1382{\cdot}4} \text{ gallons.}\end{aligned}$$

86. Metric Measure of Volume. By reasoning similar to that of Art. 85, it will be seen that a cube each of whose edges is one centimetre in length contains 10^3 or 1000 cubic millimetres. Hence the following table :

Metric Measure of Volume.

10^3 or 1000 cubic millimetres = 1 cubic centimetre (c.c.)
10^3 or 1000 „ centimetres = 1 „ decimetre
10^3 or 1000 „ decimetres = 1 „ metre

A cubic decimetre is called a **litre**, so that

1 litre = 1000 cubic centimetres.

A cubic metre is sometimes called a **stere**.

Ex. 82. *Taking one inch to be equivalent to* 2·54 *cm., find the number of cubic centimetres equivalent to a cubic inch. Use the result to determine the equivalent of a pint in litres, correct to three places of decimals, taking* 1 *gallon as* 277·274 *cub. in.*

Here we have first to find the volume of an inch cube in c.c. Hence, since 1 in. = 2·54 cm.,

$\therefore$ 1 cub. in. = 2·54 × 2·54 × 2·54 c.cm. = 16·387 c.c. correctly to three places of decimals.

$\therefore$ **1 cub. in. is equivalent to 16·387 c.c.**

Now 1 gallon = 277·274 cub. in.,

so that 1 pint = 277·274 ÷ 8 = 34·659 cub. in.
= 34·659 × 16·387 c.c.
= 567·80 c.c.,

i.e. 1 pint = 568 c.c. to the nearest c.c., and since

1000 c.c. = 1 litre,

$\therefore$ **1 pint is equivalent to 0·568 litre,**

and **1 litre is equivalent to** $\dfrac{1}{0{\cdot}568} = \mathbf{1{\cdot}76}$ **pints.**

87. Volume of a Rectangular Solid. A solid bounded by three pairs of equal, parallel and rectangular faces is called a **rectangular solid.** Such a solid is shewn in Fig. 69; the faces ABCD, EFGH

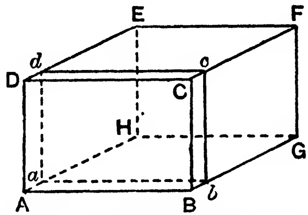


FIG. 69.—Volume of a rectangular solid.

are equal, parallel and rectangular. So also are the faces ABGH, CDEF, and the faces BCFG, ADEH. Further, since each face is a rectangle, the three pairs are mutually perpendicular to each other. The length of any edge may be taken as the **length** of the solid, and the lengths of the two perpendicular edges are then called the **breadth** and **height** of the solid respectively. Thus in Fig. 69, we may take CF as the length, CD as the breadth and BC as the height.

When the length, breadth and height are all equal so that each face is a square, the solid is a **cube.** The cube is thus a particular form of a rectangular solid, and for this reason, when length, breadth and height are not equal, the solid is often called a **cuboid.**

PROBLEM 23. *The length, breadth and height of a rectangular solid or cuboid are l, b, and h units respectively; find its volume.*

Referring again to Fig. 69, let a slice ABCD$dcba$ of unit thickness Aa be cut from the solid, then it will be clear that the number of unit cubes contained in this slice is equal to the number of unit squares in the rectangle ABCD. Thus if AB, BC contain b, h units of length respectively, then the area of ABCD $= bh$ square units; *i.e.* the slice contains bh unit cubes.

Now the number of slices each equal to ABCD$dcba$ that can be cut from the whole solid is equal to the number of unit lengths in CF, *i.e.* b.

Hence the number of unit cubes in the whole solid $= bhl$.

This is, therefore, the required volume, so that

Volume of cuboid = bhl = area of end × length.

Ex. 83. *The internal measurements of a rectangular cistern are : length* 2·4 *m., breadth* 1·22 *m., height* 1 *m. How many litres of water will it hold?* (C.S.)

Expressing the dimensions in cm., we have, length = 240 cm., breadth = 122 cm., height = 100 cm.

$\therefore$ Internal volume of cistern $= 240 \times 122 \times 100$ c.c.
$= 2{,}928{,}000$ c.c.

And since 1000 c.c. = 1 litre ;

$\therefore$ the cistern will hold **2928** litres.

Ex. 84. *A rectangular reservoir is to be constructed* 8 *ft.* 5 *in. long by* 6 *ft.* 9 *in. wide. How deep must it be made in order that it may hold* 2727 *gallons of water, taking* $277\frac{1}{4}$ *cub. in. as the equivalent of* 1 *gallon?*

Let the required depth be h in., then the internal volume of reservoir $= 101 \times 81 \times h$ cub. in.

But 1 gallon of water occupies $277\frac{1}{4}$ cub. in.

$\therefore$ 2727 gallons of water occupy $277\frac{1}{4} \times 2727$ cub. in.

This, therefore, must be the internal volume, so that

$$101 \times 81 \times h = 277\tfrac{1}{4} \times 2727 ;$$

$$\therefore\ h = \frac{1109 \times 2727}{4 \times 101 \times 81} = 92{\cdot}4 \text{ in.},$$

i.e. the depth of the reservoir must be 7 ft. 8·4 in.

88. Diagonals of a Cuboid. Let ACEG (Fig. 70) be a rectangular solid or cuboid, then, in addition to the diagonals of each rectangular face, there are also four diagonals joining opposite corners of the solid; these are the lines joining C, H; F, A; E, B; D, G. The lengths of these diagonals are easily found.

PROBLEM 24. *To find the lengths of the diagonals of a cuboid whose length, breadth and height contains l, b, and h units respectively.*

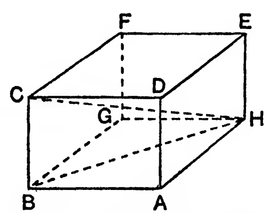


FIG. 70.—Diagonals of a cuboid.

In Fig. 70, let $CF = l$, $CD = b$, and $CB = h$, then if HB and HC be joined, we have to find the length of CH.

Now CHB is a right-angled triangle, since $\angle CBH = 90°$;

$$\therefore\ CH^2 = HB^2 + BC^2 = BG^2 + GH^2 + BC^2,$$

since HB is the diagonal of the rectangle ABGH.

And $\qquad BG = CF = l, \quad GH = FE = CD = b,$

$$\therefore \; CH^2 = l^2 + b^2 + h^2,$$

and similarly for each of the other diagonals, FA, EB, DG.

Hence the **diagonals of a cuboid are all equal, and the square on each one = sum of the squares on the three perpendicular edges.**

Ex. 85. *Find the length of the diagonal of a cuboid whose dimensions are 7 ft. 3 in., 7 ft., and 6 ft. 8 in.*

Reducing the dimensions to inches, and applying the above rule, we have:

square on diagonal

$$= 87^2 + 84^2 + 80^2 = 7569 + 7056 + 6400 = 21025 \text{ sq. in.}$$

$$\therefore \text{ Length of diagonal} = \sqrt{21025} = 145 \text{ in.} = \mathbf{12} \textbf{ ft. } \mathbf{1} \textbf{ in.}$$

89. Solids of Uniform Section. A solid whose section, cut perpendicular to its length, is always the same both in shape and size is called a **solid of uniform section.** When the end faces are also perpendicular to the length, the solid is called a **right solid of uniform section.** Let us see how the volumes of such solids may be determined.

PROBLEM 25. *To find a rule for determining the volume of a right solid of uniform section.*

Let ABCD (Fig. 71) be any right solid of uniform section, and suppose a slice AadD of unit thickness be cut off; then the number of unit cubes contained in this slice will be equal to the number of units of area in the face AD, *i.e.* in the section. Even if the face does not contain a whole number of units of area, any fraction of a unit will be the same fraction of a unit cube, since the whole slice is of unit thickness. Hence, in every case, the area of the section is a numerical measure of the volume of the slice. The whole solid can be cut into as many slices of unit thickness as there are units in its length;

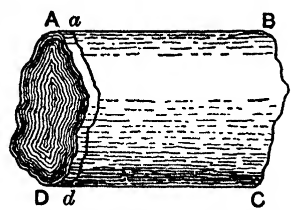


FIG. 71.—Solid of uniform section.

$\therefore$ **Volume of a right solid of uniform section is measured by the product area of section × length of solid.**

90. Prisms and Cylinders. When the end faces of a solid of uniform section are bounded by straight lines the solid is usually called a **prism**; when they are bounded by curved lines the solid is called a **cylinder.** When the end faces are perpendicular to the length of the solid, we have a **right prism** or a **right cylinder.** When the faces are parallel but not perpendicular to the length, the solid is known as an **oblique prism** or an **oblique cylinder** as the case may be.

The shape of the section of a prism gives it its name; thus when the section is a triangle, the solid is a **triangular prism,** when a hexagon, the solid is a **hexagonal prism,** and so on.

In all cases of right prisms and cylinders the rule given in Problem 25 applies, since these are particular forms of right solids of uniform section.

Ex. 86. *A cylindrical tank standing on its circular base is* 7·5 *ft. in diameter and holds* 2475 *gallons of water. Find the height of the water, taking* 6·25 *gallons to a cubic foot and* $\pi = \frac{22}{7}$.

Let h ft. = height of water; then the water in the tank takes the shape of a right circular cylinder whose base diameter is 7·5 ft., and height h ft.

$\therefore$ Its volume = area of base × height

$$= \pi \times 3{\cdot}75 \times 3{\cdot}75 \times h \text{ cub. ft.}$$

$$= \pi \times 3{\cdot}75 \times 3{\cdot}75 \times h \times 6{\cdot}25 \text{ gallons.}$$

But this must be 2475 gallons;

$$\therefore \pi \times 3{\cdot}75 \times 3{\cdot}75 \times h \times 6{\cdot}25 = 2475;$$

$$\therefore h = \frac{2475}{\pi \times 3{\cdot}75 \times 3{\cdot}75 \times 6{\cdot}25} \text{ ft.} = 8{\cdot}96 \text{ ft.}$$

Ex. 87. *A hollow circular cylinder stands on a solid square slab of the same material* 3·4 *ft. thick, whose edge is* 15·7 *ft. The internal and external diameters of the cylinder are* 8·7 *ft. and* 7 *ft. respectively. and the weight of the cylinder is equal to that of the slab. Find its height, taking* $\pi = 3{\cdot}14$.

Let the height of the cylinder be h ft., then since it is hollow, its area of section $= \pi(4{\cdot}35)^2 - \pi(3{\cdot}5)^2 = \pi \times 7{\cdot}85 \times 0{\cdot}85$ sq. ft.

$\therefore$ Its volume $= \pi \times 7{\cdot}85 \times 0{\cdot}85 \times h$ cub. ft.

Also, volume of slab $= 15{\cdot}7 \times 15{\cdot}7 \times 3{\cdot}4$ cub. ft.

Now since the weights of the cylinder and slab are equal, and they are made of the same material,

$\therefore$ their volumes must be equal, so that

$$\pi \times 7{\cdot}85 \times 0{\cdot}85 \times h = 15{\cdot}7 \times 15{\cdot}7 \times 3{\cdot}4\,;$$

$$\therefore\ h = \frac{15{\cdot}7 \times 15{\cdot}7 \times 3{\cdot}4}{3{\cdot}14 \times 7{\cdot}85 \times 0{\cdot}85} = 40 \text{ ft.}$$

91. Volume of an Oblique Prism. The essential characteristic of an oblique prism or cylinder is the fact that its end faces are parallel but not perpendicular to its length. The volume of such a solid will now be found.

PROBLEM 26. *To find the volume of an oblique prism of length l and whose right section has an area a.*

Let ABCDEA′B′C′D′E′ (Fig. 72) be an oblique prism in which the end faces ABCDE, A′B′C′D′E′ are parallel, and the length of

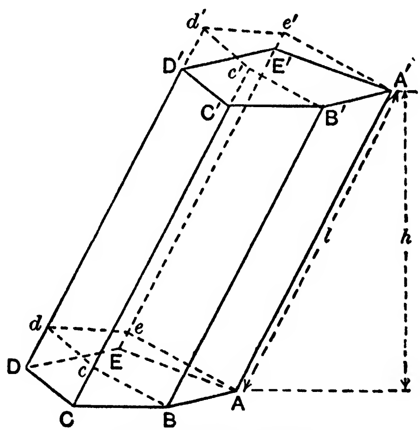


FIG 72.—An oblique prism.

whose edge AA′ $= l$. Suppose ABcde be a right section, and suppose the prism be cut through this section; then since the face ABCDE is parallel to the face A′B′C′D′E′, therefore both these are equal, so that if the solid ABCDEedc be removed and placed on A′B′C′D′E′ so that corresponding sides coincide, then the faces A′B′$c'd'e'$, ABcde will be parallel, and therefore perpendicular to the length

of the prism. Thus a right prism has been constructed whose volume is (area of ABcde) $\times l$, *i.e.* al.

$\therefore$ Volume of ABCDEA′B′C′D′E′ = vol. of ABcdeA′B′$c'd'e'$ $= al$.

$\therefore$ Volume of an oblique prism is measured by the product area of right section $\times$ length of edge.

92. Area of Right Section of an Oblique Prism. It is not always easy to measure the right section of an oblique prism, so that the above rule is often inconvenient to use. We shall now shew, however, how the area of such a section may be found from the area of the end face of the prism.

PROBLEM 27. *To find the relation between the areas of a right section and the end face of an oblique prism.*

Let ABCDEF (Fig. 73) be the end face, and ABC′D′E′F′ a right section of an oblique prism. Draw Ee, Dd perpendicular to AB,

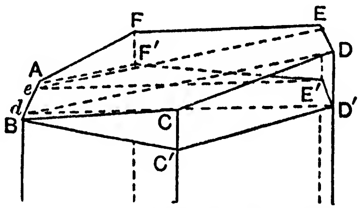


FIG. 73.—Area of a right section of an oblique cylinder.

and suppose E′e, D′d be the corresponding lines on the normal section.

Now if E′D′ be not parallel to AB, the figure BD′E′e is a trapezium, and if w be its width, then its area $= \frac{1}{2}w(e\mathrm{E}' + d\mathrm{D}')$.

Let the planes of the faces be inclined at an angle θ, then

$$\angle \mathrm{D}'d\mathrm{D} = \angle \mathrm{E}'e\mathrm{E} = \theta, \text{ and } e\mathrm{E}' = e\mathrm{E}\cos\theta,\ d\mathrm{D}' = d\mathrm{D}\cos\theta;$$

$$\therefore \text{ Area of } d\mathrm{D}'\mathrm{E}'e = \tfrac{1}{2}w(e\mathrm{E}' + d\mathrm{D}') = \tfrac{1}{2}w(e\mathrm{E} + d\mathrm{D})\cos\theta.$$

But area of $d\mathrm{DE}e = \frac{1}{2}w(e\mathrm{E} + d\mathrm{D})$, since $w = de$;

$$\therefore \text{ Area of } d\mathrm{D}'\mathrm{E}'e = (\text{area of } d\mathrm{DE}e) \times \cos\theta.$$

The same relation may similarly be shewn for every pair of corresponding areas in the two faces; it is therefore true for the whole areas, hence

Area of right section of an oblique prism is measured by the product (area of end face) $\times \cos\theta$, where θ is the inclination of the end face to the right section.

Ex. 88. *The lengths of the bounding edges of the end face of an oblique triangular prism are 4 ft. 3 in., 4 ft. 4 in. and 4 ft. 5 in. respectively, and the face is inclined at 36° 52′ to the plane of a right section ; find the area of the right section in sq. ft.*

The area of the end face = area of a triangle of sides 51 in., 52 in. and 53 in. respectively.

The semi-perimeter $s = \frac{1}{2}(51 + 52 + 53) = 78$ in.

Hence from Problem 6, area of end face

$$= \sqrt{78 \times 27 \times 26 \times 25} = \sqrt{13^2 \times 18^2 \times 5^2} = 13 \times 18 \times 5 \text{ sq. in.}$$

And from Prob. 27,

$$\begin{aligned} \text{area of right section} &= 13 \times 18 \times 5 \times \cos 36° 52' \\ &= 13 \times 18 \times 5 \times 0{\cdot}8 \text{ sq. in.} \\ &= \frac{13 \times 18 \times 5 \times 8}{144 \times 10} = \mathbf{6{\cdot}5} \text{ sq. ft.} \end{aligned}$$

93. Volume of Oblique Prism in terms of the Area of its End Face and its Vertical Height. We can now find a more convenient expression for the volume of an oblique prism by means of the result of Problem 27.

PROBLEM 28. *To find the volume of an oblique prism of vertical height h, each of whose end faces has an area a.*

Let ABCC′B′A′ (Fig. 74) be the elevation of an oblique prism

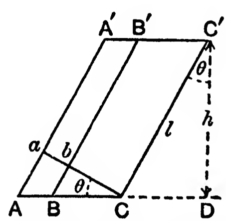


FIG. 74.—Volume of an oblique prism.

standing on one of its end faces ABC ; then the perpendicular distance CD between the two parallel end faces is called the **vertical height** of the prism. Thus $DC' = h$. Let the length of the prism $CC' = l$, and θ be the angle ACa between the planes of the end face and a normal section.

Then $\angle DC'C = 90^\circ - \angle DCC' = 90^\circ - (180^\circ - \angle ACC')$

$$= 90^\circ - (180^\circ - 90^\circ - \theta) = \theta.$$

Now by Prob. 26,

$$\text{volume of prism} = (\text{area of right section } Cba) \times l.$$

But by Prob. 27,

$$\text{area of right section} = (\text{area of end face}) \times \cos\theta = a\cos\theta,$$

and from triangle DCC′,

$$\cos\theta = h/l;$$

$$\therefore \text{ volume of prism} = al\cos\theta = ah.$$

$\therefore$ **volume of an oblique prism is measured by the product (area of one of the end faces) × (perpendicular distance between those faces).**

From the results of Problems 25 and 28, we may deduce the important fact that **prisms standing on equal bases and of the same vertical height are equal in volume.**

Further, the result of Prob. 26 applies equally to an oblique cylinder, and by dividing a plane curved area into very small elements, it may be shewn that the result of Prob. 27, and therefore that of Prob. 28, remain true for a cylinder; hence

Cylinders standing on equal bases and of the same vertical height are equal in volume.

Ex. 89. *An oblique prism has each of its end faces in the shape of a quadrilateral* ABCD *in which* AB = 2 *ft.* 2 *in.*, BC = 2 *ft.* 6 *in.*, CD = 2 *ft.* 1 *in.*, DA = 1 *ft.* 5 *in. and* CA = 2 *ft.* 4 *in. The perpendicular distance between the faces is* 6 *ft.*; *find the volume of the prism in cub. ft.*

We must first determine the area of the quadrilateral base.

The quadrilateral should be drawn, when it will be seen to consist of the two triangles ABC, ADC whose sides are 26, 30, 28 in. and 25, 17, 28 in. respectively. The semiperimeters are thus 42 and 35 in.

$$\therefore \triangle ABC = \sqrt{42 \times 16 \times 12 \times 14} = 336 \text{ sq. in.}$$

$$\triangle ADC = \sqrt{35 \times 18 \times 10 \times 7} = 210 \quad ,,$$

$$\therefore \text{ Area of end face} = 336 + 210 = 546 \text{ sq. in.}$$

$$\therefore \text{ Volume of prism} = 546 \times 72 \text{ cub. in.}$$

$$= \frac{546 \times 72}{1728} = 22{\cdot}75 \text{ cub. ft.}$$

EXERCISES 13.

Find the volume of each of the following rectangular solids whose dimensions are given:

1. 4·62 in. by 5·78 in. by 8·25 in.
2. 9·87 cm. by 6·85 cm. by 12·31 cm.
3. 7·2 in. by 8·4 in. by 4 ft. in cub. ft.
4. 15·6 in. by 16·2 in. by 26·4 in. in cub. ft.
5. 27·7 cm. by 32·5 cm. by 51·8 cm. in cub. ft.
6. 47·8 in. by 36·7 in. by 62·5 in cub. ft.
7. 23·6 cm. by 32·4 cm. by 75·8 cm. in cub. ft.
8. 48·7 cm. by 59·7 cm. by 82·7 cm. in steres.

9. Assuming that 1 cubic inch is equal to 16·387 cubic centimetres, find, to the nearest whole number, the number of cubic inches in a litre. (C.S.)

10. Assuming that a gallon of water weighs 10 lb. and that a cubic foot of water weighs 1000 oz., find, to the nearest gallon, the number of gallons of water in a rectangular tank 3 feet long, 2 feet broad, the height of the water in the tank being 2 feet 7 inches. (C.S.)

11. A rectangular tank is 4 ft. long, 3 ft. wide and 2 ft. 6 in. deep. Calculate the number of gallons and the weight of water it will hold. A cubic foot of water weighs 62·5 lb. and a cubic foot contains 6·25 gallons. (U.L.C.I.)

12. Calculate the third dimension of a rectangular cistern to hold 100 gallons, the other two dimensions being 2 ft. 6 in. and 3 ft. 4 in. A cubic foot of water contains 6·25 gallons. (U.L.C.I.)

13. If $\frac{1}{5}$th of the air is oxygen, how many cubic feet of oxygen are there in a room which measures 20 ft. × 24 ft. × 11 ft. 6 in.? (U.L.C.I.)

14. A rectangular tank, whose internal dimensions are 1·6 metres long, 1·5 metres wide, and 0·4 metre deep, is half full of water. Find the depth of the water after 300 cubic decimetres of water have been poured in. (C.S.)

15. A rectangular block of iron of square section has weight 65 lb. and thickness 2·6 in. Find the side of the square section being given that the iron weighs 444 lb. per cubic foot. (L.M.)

16. A rectangular block of brass measures 10 ft. 3 in. by 7 ft. 4 in. by 3 ft. 6 in., and weighs 62·75 tons. Find the weight in lb. of one cubic inch.

17. A square plate of iron of edge 192 cm. and thickness 12 cm. weighs 3354 kilograms. Find the weight in grams of one cubic centimetre.

18. Shew where the mistake is in the following, and give the true volume in cubic centimetres correctly to one place of decimals:

$$\text{Volume of rectangular block} = 4{\cdot}5 \times 3{\cdot}75 \times 2{\cdot}25 \text{ cu. in.}$$
$$= 4{\cdot}5 \times 3{\cdot}75 \times 2{\cdot}25 \times 2{\cdot}54 \text{ c.c.}$$

19. Estimate the number of tons of water in a pond of which the surface is an acre in extent and the average depth is 10 ft. A cubic foot of water weighs 1000 oz. (C.P.)

20. A rectangular box without a lid, whose internal dimensions are 0·745 m. long by 0·5 m. wide by 0·2 m. deep is made out of wood 2 cm. thick. Find its weight when empty to the nearest kg. if a cubic centimetre of wood weighs 0·8 of a gram. (L.S.)

21. A rectangular tank whose internal dimensions are 2 ft. 11 in. by 2 ft. 4 in. by 1 ft. 9 in. is quite full of water when a metal cube is dropped in, and as a consequence one-sixtieth of the water overflows. Find the length of an edge of the cube.

22. A water tank is 3 ft. deep, and its base is a rectangle 4 ft. 6 in. by 2 ft. 9 in., inside measurements. Find how much the water will rise if a block of stone measuring 6 in. by $5\frac{1}{2}$ in. by $4\frac{1}{2}$ in. is placed in the tank when it is half full of water. (C.S.)

23. The external dimensions of a closed rectangular cistern are 3 ft. 6 in., 2 ft. 9 in., 2 ft. 3 in.; and the thickness of the material is $\frac{5}{8}$ of an inch. How many gallons of water will it hold? 1 gallon of water weighs 10 lb., 1 cub. ft. of water weighs 1000 oz. (L.M.)

24. Find, to the nearest cubic inch, the volume of a quantity of lead weighing 20 kilograms. 1 c.c. of lead weighs 11·37 gm., 1 metre equals 39·37 in. (L.M.)

25. Calculate the number of bricks required for building the walls of a rectangular barn 40 ft. long by 20 ft. wide externally, neglecting the space taken up by mortar between the bricks. The walls are 15 ft. high and 6 in. thick, each brick is 6 in. by 3 in. by 2 in.; and a gap 12 ft. wide by 10 ft. high is left for a door in the centre of one of the walls. (L.M.)

26. The base of a reservoir is a rectangle 100 yards long by 50 yards wide. The two short sides are vertical and the two long sides are inclined outwards at an angle of 45° with the vertical. Find the volume of the water in gallons which will fill the reservoir to a depth of 6 ft., taking a gallon as equal to $277\frac{1}{4}$ cub. in. (L.M.)

27. Find the length of the diagonal of a cuboid measuring 3·2 ft. by 5·1 ft. by 6 ft.

28. A room is 8 yd. long, 5 yd. wide and 10 ft. high. Calculate the length of string which, tightly stretched, would reach from the centre of the ceiling to a corner of the floor. Give the answer to the nearest inch. (C.P.)

29. The diagonal of a cuboid is 2·6 times the diagonal of one face measuring 7 in. by 2 ft. Find the third dimension of the solid.

30. The length of a cuboid is greater than twice its width by six inches, and the height is greater than five times the width by five inches. The diagonal is 7 ft. 5 in. long; find the dimensions of the solid.

31. A right triangular prism is 6 ft. long, and the sides of its base are 13 in., 37 in. and 40 in. respectively. Find its volume in cubic feet.

32. Give the volume and surface of a cylinder whose diameter is 30 inches and length 45 inches. (U.L.C.I.)

33. Find the volume of a cylinder 3·8 cm. long and 1·58 cm. in diameter.

34. What is the volume of a cylinder 1·26 cm. in diameter and 5·12 cm. long?

35. How many litres will a cylindrical vessel 11 cm. in diameter and 15·78 cm. in height contain?

36. A piece of brass eight centimetres long in the shape of a right circular cylinder displaces 1413 c.c. of water, and its diameter is 15 cm. Deduce the value of π.

37. A circular pit, 18 feet in diameter, is sunk to a depth of 200 yards. Taking the average weight of one cubic foot of material as 138 lb., find the total weight in tons of rock excavated. (U.L.C.I.)

38. A trough of semi-circular section with stopped ends is 8 feet long and 12 inches wide. Calculate the number of gallons it will hold, taking $277\frac{1}{4}$ cubic inches to one gallon. (U.L.C.I.)

39. The volume of a disc, 8 mm. thick, is calculated on the assumption that its diameter is 10·4 cm. If the correct length of the diameter is 10·36 cm., find the percentage error in the calculated volume. (N.U.T.)

40. A cylindrical vessel of height = diameter of base = 85 cm. contains water up to a height of 35 cm. How much will the water rise when an iron cube of edge 25 cm. is placed inside the vessel? (N.U.T.)

41. A stone garden-roller which has a length of 3 ft. 6 in., has a volume of 18 cu. ft. 609 cub. in.; what is the diameter of its circular ends? Take $\pi = \frac{22}{7}$. (J.M.B.)

42. Find the weight of a circular iron disc one foot in diameter and one inch thick, punched with 12 circular rivet holes each $\frac{1}{2}$ in. in diameter, 1 cu. ft. of iron weighing 480 lb. Give the answer to the nearest quarter lb. (L.S.)

43. The practical approximate rule for calculating the number of gallons in a cylindrical vessel is to multiply the square of the diameter in inches by the depth of the water in yards, and divide by 10. Apply

this rule to find the number of gallons in a well 5 ft. in diameter, the water being 15 ft. deep.

Calculate also the true quantity, taking 6·23 gallons to the cubic foot, and estimate to one significant figure the percentage error in the approximation. (D.U.)

44. Rain falls on 500 square feet of roof to a depth of half an inch. It runs off into a circular tank 5 ft. in diameter. Find, to the nearest inch, the height of the water in the tank. (C.S.)

45. A circular fish pond 21 yards in diameter with a flat bottom is to be made in the centre of a square courtyard whose side is 63 yards. To save the trouble of removal, the earth excavated is spread evenly over the remaining surface of the courtyard. Find in inches how much the level of the courtyard has been raised when the bottom of the pond is 5 yards below the new level of the courtyard, taking π as $\frac{22}{7}$.

46. In a hollow cylinder, open at both ends, there are 678·24 cu. in. of material. Its external diameter is 1 ft. 2 in. and its length is 9 in. Find the thickness of its walls, taking $\pi = 3{\cdot}14$.

47. A hollow cylinder of lead has an external diameter of 42 cm. and an internal diameter of 32 cm. Its length is 95 cm., and lead weighs 11·28 gr. per c.c. Calculate the weight of the cylinder in kilograms.

48. A cylindrical shaft 30 feet long is to have its curved surface covered with concrete 6 inches thick. If the internal diameter of the concrete when completed is 5 ft. 3 in., find the weight in tons of the concrete used. One cubic foot of concrete weighs 195 lb. (J.M.B.)

49. A hollow cylinder whose internal and external radii are $3\frac{1}{2}$ ft. and 9 ft. respectively, has the same volume as a cube whose edge is 11 ft. Calculate the length of the cylinder, taking $\pi = 3\frac{1}{7}$.

50. A mile of copper wire weighs 7700 lb., and a cubic foot of copper weighs 540 lb. Find the diameter of the wire in inches, taking the volume of wire to be $0{\cdot}7854 \times \text{length} \times (\text{diameter})^2$. (L.S.)

51. An inch of rain on an acre weighs M tons. A centimetre of rain on a hectare weighs N tons. Given that 1000 cubic centimetres of water weigh a kilogram, a cubic foot of water 62·4 lb., and that 1000 kilograms $= 0{\cdot}984$ of a ton, shew that M tons exceeds N tons by nearly 2 tons 14·5 cwt. (L.S.)

52. Taking a cubic foot of water to weigh 1000 oz. Av., what must be the area of the base of a cylindrical vessel 10 in. deep to contain a pound and a half of water, and what will be the diameter of the base? (L.S.)

53. Find the volume of an oblique triangular prism whose perpendicular height is 25 cm., and whose end faces have sides of lengths 14·5 cm., 13·7 cm. and 18·8 cm.

CHAPTER XIV

VOLUMES OF PYRAMIDS AND CONES

94. Pyramid. A solid standing on a plane rectilineal base and having plane triangular faces which meet in a common vertex is called a **Pyramid.** According to the shape of the base, so the pyramid is named. A triangular pyramid is called a **tetrahedron.**

95. Area of a Section of a Pyramid parallel to the Base. It is important to know how the sectional area of a pyramid varies at different distances from the base. This will be our first problem on the pyramid.

PROBLEM 29. *To find how the area of a section of a pyramid parallel to the base varies with its distance from the vertex.*

Let PQRSV (Fig. 75*a*) be a pyramid of height h, standing on a square base PQRS, each of whose sides is a units long.

Let $pqrs$ be a plane section parallel to the base and at a distance y units from V. If the perpendicular from V to the base meets $pqrs$ in c and the base in C, then $\mathrm{V}c = y$, and $\mathrm{VC} = h$.

Now by similar triangles,

$$\frac{pq}{\mathrm{PQ}} = \frac{\mathrm{V}p}{\mathrm{VP}} = \frac{\mathrm{V}c}{\mathrm{VC}} = \frac{y}{h};$$

$$\therefore\ pq = y\,.\,\mathrm{PQ}/h = ya/h.$$

Similarly, we have $\quad qr = ya/h.$

$$\therefore\ \text{Area of } pqrs = pq \times qr = y^2a^2/h^2.$$

But area of base $\mathrm{PQRS} = a^2$;

$$\therefore\ \frac{\text{area of } pqrs}{\text{area of PQRS}} = \frac{y^2}{h^2},$$

or, we might say that since the area of the base and the height of the pyramid remain the same whatever the section $pqrs$ may

be, the ratio a^2/h^2 is constant for the pyramid. If, therefore, we denote this constant ratio by k, so that $k=a^2/h^2$, then,

$$\text{area of section pqrs} = \mathbf{k} \cdot \mathbf{y}^2,$$

i.e., the **area of any section parallel to the base is proportional to the square of its distance from the vertex.**

For a pyramid standing on any base it is easy to see that by dividing the base and any parallel section into corresponding unit squares however small, the above fact is still true; hence it is true of all pyramids.

Ex. 90. *The sides of the triangular base of a tetrahedron of height 4 ft. 4 in. are 2 ft. 1 in., 4 ft. 3 in., 4 ft. 4 in. long respectively. It is cut by a plane parallel to the base and 2 ft. 8·5 in. from it; find the area of the section thus made.*

The semi-perimeter of the base $=\frac{1}{2}(25+51+52)=64$ in.;

$\therefore$ Area of base $=\sqrt{64\times 12\times 13\times 39}=8\times 6\times 13$ sq. in.

Let a sq. in. be the area of the section; its distance from the vertex is $52-32{\cdot}5=19{\cdot}5$ in.;

$$\therefore\ a=\frac{19{\cdot}5\times 19{\cdot}5}{52\times 52}\times 8\times 6\times 13 \text{ sq. in.}$$

$$=\frac{8\times 6\times 13\times 39\times 39}{52\times 52\times 2\times 2}=87{\cdot}75 \text{ sq. in.}$$

96. Pyramids on Equal Bases and of the same Height. It will now be shewn that the volumes of such pyramids are equal.

PROBLEM 30. *To find the relation between the volumes of two pyramids of the same height and standing on bases of equal area.*

Let $VPQRS$, $V_1P_1Q_1R_1S_1$ (Fig. 75) be two pyramids of height h and whose bases $PQRS$, $P_1Q_1R_1S_1$ each have area a. Let $pqrs$, $p_1q_1r_1s_1$ be two thin slices parallel to their respective bases and each distant y from its vertex; then from Prob. 29,

$$\text{area of } pqrs = ay^2/h^2, \text{ and area of } p_1q_1r_1s_1 = ay^2/h^2,$$

i.e. $\qquad$ area $pqrs$ = area $p_1q_1r_1s_1$,

and if the thicknesses of the slices are equal, and very small, the volumes of the slices will be approximately equal. Indeed, as we make the common thickness smaller and smaller, the volumes will become more nearly equal, so that when the thickness be indefinitely diminished, the volumes will ultimately become equal. The same is true of all corresponding slices making up the

pyramids, so that the sum of the volumes of all slices in one pyramid is equal to the sum of all slices in the other pyramid:

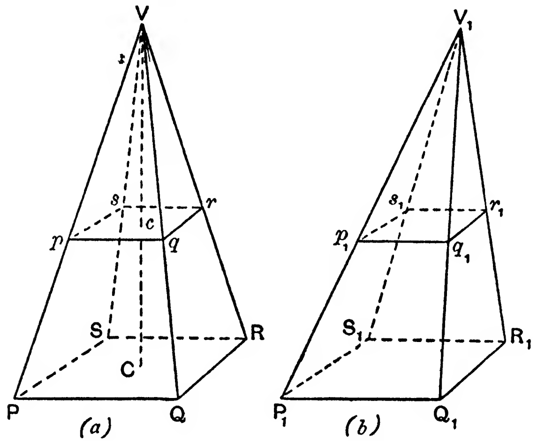


FIG. 75.—Right and oblique pyramids.

hence the **volumes of pyramids of equal altitude standing upon bases of equal area are equal.**

97. Volume of a Pyramid. We can now find a rule for determining the volume of a pyramid.

PROBLEM 31. *To find the volume of a pyramid.*

We shall solve this problem by two independent methods.

(i) Let PABCQ (Fig. 76) be a triangular prism; then we can cut this prism into three pyramids SABC, PASQ, QCAS as indicated. Of these, the pyramids SABC, PASQ are clearly of the same height, and their bases ABC, PSQ are equal; hence they are equal in volume. Further, the pyramids PASQ, SAQC have equal bases PAQ, QAC since they are halves of the uncut face PQCA; and the common vertex S is the same distance from these bases, so that the pyramids are equal in volume. Hence the three pyramids

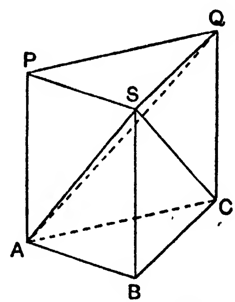


FIG. 76.—Volume of a pyramid.

SABC, PASQ, QCAS are equal in volume, so that the volume of each is one-third that of the prism.

In the case of a pyramid whose base is a quadrilateral or polygon, these figures may be divided into triangles however small, and the pyramid cut so that each smaller pyramid has one triangle for its base. The volume of each of these pyramids is $\frac{1}{3}$ that of the prism standing on the same base and of the same height, so that on summing, the volume of the whole pyramid is $\frac{1}{3}$ the corresponding prism. Hence, in all cases, the **volume of a pyramid $= \frac{1}{3}$ volume of a right prism standing on the same base and of the same altitude;**

i.e. **volume of pyramid is measured by the product $\frac{1}{3}$ (area of base) $\times$ altitude.**

(ii) Referring to Fig. 75*a*, let the base area PQRS be a, and the height VC $= h$.

Let VC be divided into n equal parts, where n is a large whole number; then the length of each part $= h/n$, which is very small.

Now suppose at a distance of r of these divisions from V, *i.e.* rh/n, we have a slice $pqrs$ parallel to the base and whose thickness is one of the equal divisions, *i.e.* h/n; then the approximate volume of slice $=$ area of $pqrs \times$ thickness $= \frac{a}{h^2} \cdot \left(\frac{rh}{n}\right)^2 \cdot \frac{h}{n} = \frac{ar^2h}{n^3}$. If we give r all the values 1, 2, 3, ... n in turn, we shall obtain the volumes of all the slices, so that the approximate volume of the pyramid $= \frac{ah}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$.

But, by algebra, we know that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \tfrac{1}{6}n(n+1)(2n+1);$$

$\therefore$ Approximate volume of pyramid

$$= \tfrac{1}{6}ah(n+1)(2n+1)/n^2$$
$$= \tfrac{1}{6}ah(2 + 3/n + 1/n^2).$$

Now as n is made larger and larger, the more nearly does this expression represent the volume, so that as n is indefinitely increased, $3/n$ and $1/n^2$ tend to zero, and ultimately we get

$$\text{volume of pyramid} = \tfrac{1}{6}ah \,.\, 2 = \tfrac{1}{3}ah,$$

i.e. $\frac{1}{3}$ of the volume of the right prism whose end face has an area a, and whose altitude is h.

This method is quite independent of the fact that pyramids on equal bases and of the same altitude are equal in volume, and such fact may therefore be immediately deduced from it.

Ex. 91. *Find the weight in tons of a solid rectangular pyramid of bronze* 15 *ft. high, whose base measures* 7 *ft. by* 5 *ft.* 4 *in., the weight of the bronze being* 546 *lb. per cubic foot.*

The volume of the pyramid $=\frac{1}{3}$(area of base) $\times$ height

$$=\tfrac{1}{3}\times 7\times 5\tfrac{1}{3}\times 15 \text{ cub. ft.}$$

$\therefore$ Its weight $=\frac{1}{3}\times 7\times 5\frac{1}{3}\times 15\times 546$ lb.

$$=\frac{7\times 16\times 15\times 546}{3\times 3\times 112\times 20}\text{ tons}=\mathbf{45{\cdot}5}\text{ tons.}$$

98. Regular Pyramids. When the lateral faces of a pyramid are all equal isosceles triangles so that the base is a regular rectilineal figure, the pyramid is said to be **regular.** The line drawn from the vertex perpendicular to the base of such a solid is called its **axis**, and clearly the axis meets the base in its middle point, which is the centre of the circle circumscribing the regular rectilineal figure.

Ex. 92. *A regular pyramid stands on a hexagonal base whose side is* 2 *ft.* 4 *in. long, and one of its equal lateral edges is* 4 *ft.* 5 *in. long. Find the volume of the pyramid in cubic feet.*

Let VABCDEF (Fig. 77) be the pyramid standing on the hexagonal base ABCDEF. Let OV be the axis, then O is the centre of the circle circumscribing the hexagon, so that

$$OA=OB=AB=28 \text{ in.}$$

Hence OAB is an equilateral triangle, and its area $=196\sqrt{3}$ sq. in.

And the hexagon consists of six equilateral triangles each equal in area to OAB;

$\therefore$ area of base $=6\times 196\sqrt{3}$ sq. in.

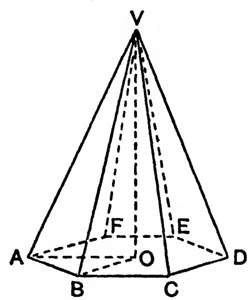


FIG. 77.—Regular hexagonal pyramid.

We have next to find the height OV.

Since the pyramid is regular, VOA is a right angle,

$$\therefore\ OV^2=AV^2-OA^2=53^2-28^2=81\times 25=45^2;$$

$$\therefore\ OV=45 \text{ in.}$$

$\therefore$ Volume of pyramid $=\frac{1}{3}\times 6\times 196\sqrt{3}\times 45$ cub. in.

$$=\frac{6\times 196\times 1{\cdot}732\times 45}{3\times 1728}\text{ cu. ft.}=\mathbf{17{\cdot}7}\text{ cub. ft.}$$

99. The Cone. If the number of sides in the base of a regular pyramid be increased indefinitely, we know that the base then becomes ultimately a circle. In such a case the lateral faces also become ultimately a single curved surface, and the pyramid becomes a **right circular cone.** We may, however, regard the cone as the solid formed by the revolutions of a right-angled triangle about one of the sides forming the right angle. Thus in Fig. 78, the right circular cone VPQR is generated by the revolution of the right-angled triangle VCP about the side VC, which is called the **axis** of the cone. The side CP traces out the circular **base** PQR, whilst the hypotenuse VP traces out the curved surface, and is called a **generator** of the cone.

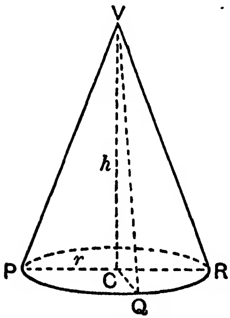


FIG. 78.—Right circular cone.

PROBLEM 32. *Find the volume of a right circular cone whose base radius is r and height h.*

Since a right circular cone is really a special form of a regular pyramid, its volume is, by Prob. 31, measured by the product

$$\tfrac{1}{3}(\textbf{area of base}) \times \textbf{height}.$$

Now the area of the base = area of a circle of radius $r = \pi r^2$;

$\therefore$ **Volume of a right circular cone whose base radius is r and height h is $\frac{1}{3}\pi r^2 h$.**

Ex. 93. *A right circular cone is generated by the revolution of a triangle* VCP, *having a right angle at* C, *about the side* VC. *If* CP = 7·8 *in. and* PV = 4 *ft.* 3 *in., find the volume of the cone in cu. ft.*

Referring to Fig. 78, it will be clear that CP is the radius of the base and CV is the height; to find CV, we have

$$CV^2 = PV^2 - PC^2 = 51^2 - 7{\cdot}8^2 = 58{\cdot}8 \times 43{\cdot}2 = 2540{\cdot}16;$$

$$\therefore\ CV = \sqrt{2540{\cdot}16} = 50{\cdot}4 \text{ in.}$$

$$\therefore\ \text{Vol. of cone} = \tfrac{1}{3}\pi \times CP^2 \times CV = \tfrac{1}{3}\pi \times 7{\cdot}8^2 \times 50{\cdot}4 \text{ cub. in.}$$

$$= \frac{22 \times 7{\cdot}8 \times 7{\cdot}8 \times 50{\cdot}4}{3 \times 7 \times 1728} = 1{\cdot}859 \text{ or } \mathbf{1{\cdot}86} \text{ cub. ft.}$$

EXERCISES 14.

Take $\pi = 3{\cdot}14$, unless otherwise stated.

1. A pyramid standing on a triangular base whose sides are 10 cm., 17 cm. and 21 cm., is cut by a plane parallel to the base at a distance 6 cm. from it. If the height of the pyramid be 30 cm., find the area of the section cut by the plane.

2. A right circular cone of height 9 in. and base radius 4·5 in. is cut by a plane parallel to the base and 5 in. from it; find the area of the section thus made.

3. A pyramid 1 ft. 10 in. high stands on a rectangular base 16 in. by 11 in., and is cut by a plane parallel to the base so that the area of the section is 68·75 sq. in. Find the distance of this plane from the base.

4. The radius of a section parallel to the base of a right circular cone and 2·1 in. from it is 5 in.; the radius of the base is 8 in., find the height of the cone.

Find the volume of each of the following pyramids or cones:

5. Base, a square of 9 ft. side; height, 5 ft. 4 in.

6. Base, a square of 4·5 ft. side; height, 8·4 ft.

7. Base, a rectangle 14·9 cm. by 13·2 cm.; height, 15·5 cm. Give the volume in cubic inches, taking 16·39 c.c. as the equivalent of 1 cub. in.

8. Triangular base whose sides measure 39 cm., 17 cm., 44 cm.; height, 74·5 cm. Give the volume in cub. in., using the equivalent of Ex. 7.

9. Triangular base whose sides measure 37 in., 13 in., 40 in.; height, 81 in. Give the volume in cubic feet.

10. Base, a quadrilateral ABCD in which AB = 45 in., BC = 20 in., CD = 34 in., DA = 39 in., BD = 42 in.; height, 43·2 in.

11. Base, a quadrilateral ABCD having AB = 7·5 cm., BC = 7·8 cm., CD = 5·8 cm., DA = 4·1 cm.; AC = 5·1 cm.; height, 7·5 cm.

12. Base, a quadrilateral ABCD having AB = 56 in., CD = 16 in., AC = 65 in., $\angle B = \angle D = 90^\circ$; height, 27 ft.

13. Circular base of radius 7·5 in.; height, 1 ft. 8 in.

14. A circular base of diameter 26·4 cm.; height, 89·4 cm. Give the volume in cub. ft., using the equivalent of Ex. 7.

15. Base, a regular hexagon of side 8 in.; height, 10 in.

16. Base, a regular octagon of side 6 in.; height, 1 ft. 8 in.

17. The base of a regular rectangular pyramid measures 63 in. by 41·6 in., and the equal slanting edges are each 84·5 in. long. Find the volume of the pyramid in cubic feet.

18. Find the height of a triangular pyramid whose base measures 51 in., 52 in., and 53 in. respectively, and whose volume is 16·25 c. ft.

19. A right circular cone, made of cast iron, weighs 62·8 tons, and its height is 12 ft. Taking one cubic foot of iron to weigh 448 lb., find the radius of its base.

20. A solid cast iron pyramid 18 ft. high standing on a square base of 5 ft. side weighs 30 tons. Find the weight of a cubic foot of cast iron.

21. A conical vessel 9 in. high holds 2·5 pints of water. Find the radius of its rim, taking 6·25 gallons to a cubic foot.

22. A square pyramid 135·7 cm. high is to be cut into two portions of equal volume by a plane parallel to the base. Find the distance of the plane from the vertex.

23. A right circular cone 1 ft. 9 in. high, whose base diameter is 1 ft., is cut by a plane parallel to its base at a point 10 in. from the base. Find the volumes of the two portions.

24. The top stone of a gate post is in the form of a square prism of 2 ft. 3 in. side and 4 in. thick, surmounted by a pyramid 6 in. high standing on a square face of the prism as base. Find the weight of the stone if the material of which it is made weighs 176 lb. per cub. ft.

25. Find the radius of the base of a right circular cone whose volume and height are equal respectively to the volume and height of a right circular cylinder of diameter 1 ft. 8 in.

26. A cone of depth 53·6 cm. and base diameter 24·7 cm. just fits into a hollow cylinder so that the vertex of the cone meets the centre of the circular base of the cylinder. Find the volume of the air-space between the cone and the cylinder. (N.U.T.)

27. A hollow copper cone is of uniform thickness throughout its base and sides, and weighs 80 lb. Its external dimensions are: diameter of base 1 ft., height 1 ft. If a cubic inch of copper weighs 0·32 lb., find the thickness of the cone. (U.L.C.I.)

28. A right circular cone of lead whose base radius is 9 in. is melted down and re-cast into a right circular cylinder whose length is equal to three-quarters of the height of the cone. Assuming no lead is wasted, find the radius of the cylinder.

29. A right circular cylinder of height 1 ft. 8 in. is inscribed in a right circular cone so that its base lies on the base of the cone and its axis along that of the cone. Find its base radius and its volume if the height and base diameter of the cone are 2 ft. 8 in. and 1 ft. 8 in. respectively.

30. A right circular cone is generated by the revolution of a triangle ABC, right-angled at C, about CA. If $AB = 18·1$ in., $BC = 1·9$ in., find the volume of the cone.

31. ABC is a triangle having BC $=8\cdot5$ cm., CA $=13\cdot2$ cm., and $\angle C=90°$. The triangle revolves about AB; find the volume of the solid thus generated.

32. A solid is formed by the revolution of an isosceles triangle ABC about its base AC. Find the volume of the solid when AB $=$ BC $=6\cdot5$ in., and AC $=3\cdot2$ in.

33. ABC is a triangle in which AB $=21\cdot2$ in., BC $=19\cdot5$ in., CA $=11\cdot3$ in., and it revolves about BC thus generating a solid. Find the volume of the solid in cub. ft.

34. A solid spindle is formed by the revolution of a quadrilateral ABCD about the side BC. If AB $=8\cdot9$ in., BC $=29\cdot6$ in., DA $=25\cdot7$ in., and AD, BC are each perpendicular to DC, find the volume of the spindle in cub. ft.

35. ABCD is a trapezium in which AD is parallel to BC, and the distance between them is 15 cm. The whole figure revolves about the side AD; find the volume of the solid thus generated in cu. in., when AD $=99\cdot1$ cm., and BC $=32\cdot4$ cm., using the equivalent given in Ex. 7.

36. A wedge is shewn in Fig. 79. PQRS is the rectangular base of a right triangular prism whose ends are cut so that the triangular faces

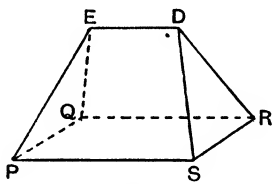


FIG. 79.

SRD, PQE are equally inclined to PQRS, and the faces EDSP, EDQR are equal trapeziums. If PS $=l$, SR $=b$, ED $=e$, and the perpendicular distance between the edge ED and the base $=h$, find an expression for the volume of the solid.

Hence calculate the volume in cub. ft. when $l=22$ in., $b=16$ in., $e=10$ in. and $h=12$ in.

CHAPTER XV

SURFACE AREA OF A SOLID.

100. Surface Area of a Prism. The total area of all the faces of a prism is called its **surface area.** When, however, the areas of the end faces are not included, the area of all the remaining faces is known as the **lateral area.**

PROBLEM 33. *To find the lateral area of* (i) *a right prism,* (ii) *an oblique prism.*

(i) The lateral faces of a right prism are all rectangles, whose length is equal to that of the prism and whose width is equal to the length of one side of the base, *i.e.* the end face.

Let l be the length of the prism, and $s_1, s_2, s_3, \ldots$ the lengths of the sides of the base:

$$\begin{aligned}\text{then the lateral area of prism} &= ls_1 + ls_2 + ls_3 + \ldots \\ &= l(s_1 + s_2 + s_3 + \ldots) \\ &= l(\text{sum of sides of base}).\end{aligned}$$

Now the sum of the sides of the base is its perimeter, and since every right section of the prism is equal in all respects to an end face, the perimeter of a right section = perimeter of end face,

$\therefore$ **The lateral area of a right prism is measured by the product, (length of prism) × (perimeter of right section).**

(ii) The lateral faces of an oblique prism are parallelograms whose length is equal to that of the prism and whose width is equal to the length of a side of a right section.

Hence, **the lateral area of an oblique prism is measured by the product (length of prism) × (perimeter of right section).**

Ex. 94. *A wooden tank 4 ft. by 3 ft. by 5 ft. deep is to be lined with sheet zinc weighing $1\frac{1}{2}$ lb. per square foot. Calculate the weight of zinc to be used, allowing 10 per cent. on the net size for laps and waste.* (U.L.C.L)

The area of the base $= 4 \times 3 = 12$ sq. ft.

Lateral area $= 2(4+3) \times 5 = 70$ sq. ft.

$\therefore$ Total area to be covered $= 82$ sq. ft.

10 per cent. of 82 sq. ft. $= 8{\cdot}2$ sq. ft.

$\therefore$ Area of zinc required $= 82 + 8{\cdot}2 = 90{\cdot}2$ sq. ft.

and weight of zinc required $= 90{\cdot}2 \times 1{\cdot}5 = \mathbf{135{\cdot}3}$ lb.

101. Lateral Area of a Cylinder. As the cylinder is really a special form of a prism, the rules found in Prob. 33 also apply.

PROBLEM 34. *Find the lateral area of a right cylinder of length l, whose base is any closed curve of perimeter s.*

Deduce the lateral area of (i) *a right circular cylinder of base radius r,* (ii) *an oblique cylinder.*

If we cut a piece of thin paper so that when it is wrapped round the curved surface of the cylinder it covers the surface completely, without overlapping, the area of the paper will be equal to that of the curved surface.

The shape of the paper, when spread out on a flat table, will be rectangular, since the end faces of the cylinder are perpendicular to its length. The width of the rectangular sheet will be equal to the perimeter of an end face, and its length will be the same as that of the cylinder; hence the

lateral area of a right cylinder is measured by the product, (length of cylinder) × (perimeter of right section),

i.e. the lateral area $= ls$.

(i) When the base is a circle of radius r, its perimeter $= 2\pi r$;

$\therefore$ **lateral area of a right circular cylinder of length l and base radius r is measured by the product $2\pi rl$.**

(ii) When the cylinder is oblique, the shape of the paper completely covering its curved surface is that of a parallelogram, whose length is that of the cylinder and whose width is equal to the perimeter of a right section, hence; the

lateral area of an oblique cylinder is measured by the product (length of cylinder) × (perimeter of right section).

Ex. 95. *A cylindrical-shaped vessel* 6 *in. in diameter and* 12 *in. long, closed at both ends, is made of copper weighing* 2 *lb. per sq. ft. Determine its weight.* (U.L.C.I.)

Here we have to find the surface area.

Now the lateral area $= 2\pi \times 3 \times 12 = 72\pi$ sq. in.

Area of the ends $= 2\pi \,.\, 3^2 = 18\pi$ sq. in.

$\therefore$ Total surface $= \pi(72+18) = 90\pi$ sq. in. $= 90\pi/144$ sq. ft.

$\therefore$ Weight of copper $= 180\pi/144 = 3{\cdot}93$ lb.

102. Lateral Area of a Pyramid. Each of the sloping faces of a pyramid is triangular in shape, so that the lateral area is the sum of the areas of these triangular faces, which are equal when the pyramid is regular, but unequal in other cases.

PROBLEM 35. *To find the lateral area of a pyramid.*

Let VPQRST (Fig. 80) be a regular pyramid standing on a polygonal base PQRST. If there are n equal sides in the base, then there will be n triangular faces all equal in area.

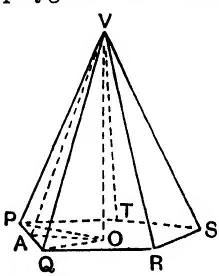


FIG. 80.—Lateral area of a pyramid.

Let VA be the perpendicular from V to PQ, then VA is called the **slant height.** Clearly the area of triangle VPQ $= \frac{1}{2}$PQ . VA, so that the **lateral area of a regular pyramid whose base contains n equal sides is measured by the product**

$$\tfrac{1}{2}n(\text{side of base}) \times (\text{slant height}).$$

When the pyramid is not regular, the area of each triangular face must be separately determined, either by measuring its three sides or its base and slant height.

Ex. 96. *Find the area of sheet lead required to cover the roof of a turret in the shape of a regular pyramid, whose sloping edge is* 12 *ft.* 4 *in. long, and whose base is a pentagon of side* 8 *ft.*

Let VPQRST (Fig. 80) represent the turret, then if VA be the slant height, since VP $= 12\frac{1}{3}$ ft. and PA $= 4$ ft., we have

$$VA^2 = VP^2 - PA^2 = \left(\tfrac{37}{3}\right)^2 - 4^2 = \tfrac{49}{3} \times \tfrac{25}{3};$$

$$\therefore\ VA = \tfrac{35}{3} \text{ ft.}$$

Hence, area of face VPQ $= \frac{1}{2}$. PQ . AV $= 4 \times \frac{35}{3}$ sq. ft.

$\therefore$ Lateral area $= 4 \times \frac{35}{3} \times 5 =$ **$233\frac{1}{3}$** sq. ft.

103. Lateral Area of a Right Circular Cone. As the right circular cone is a special form of a regular pyramid, the rule found in Prob. 35 may be applied to find the area of the curved surface. We may, however, determine the measure of the lateral area by a method similar to that used in Prob. 34 to find the curved surface of the cylinder.

PROBLEM 36. *To find the lateral area of a right circular cone of height h and base radius r.*

Let VPQ (Fig. 81*a*) be a right circular cone, whose height $OV = h$, and base radius $= OQ = r$. The length of the generator PV or QV is called the **slant height** of the cone; denote its length by l.

Now if a thin piece of paper be cut so that when wrapped round the curved surface it completely covers that surface without overlapping, then on flattening out the paper it takes the shape V′P′Q′, where $V'P' = V'Q' = VQ = l$, and the arc $P'Q'$ = circumference of the circular base PQ.

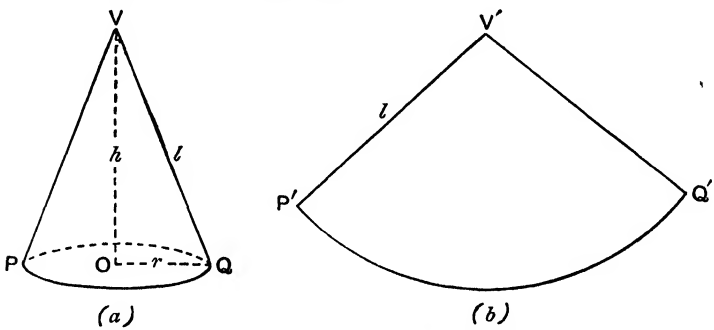


FIG. 81.—Curved surface of a right circular cone.

V′P′Q′ is, therefore, the sector of a circle, and by Prob. 14 its area is $\frac{1}{2} \cdot V'P' \times \text{arc} \cdot P'Q'$, *i.e.* $\frac{1}{2}$(slant height) × (circumference of base). Hence, the **lateral surface of a right circular cone is measured by the product,**

$$\tfrac{1}{2}(\textbf{slant height}) \times (\textbf{circumference of base}).$$

The slant height

$$l = VQ = \sqrt{QO^2 + OV^2} = \sqrt{r^2 + h^2},$$

and circumference of base $= 2\pi r$;

∴ **lateral surface of a right circular cone of height h and base radius r is** $\pi r\sqrt{r^2 + h^2}$.

Ex. 97. *A piece of thin metal in the form of a quadrant of a circle of radius* 10 *in. is bent so that the extreme radii coincide, and the metal forms an open conical cup. Find how many cubic inches of water it will hold.* (C.P.)

Let r = radius of base, h = height of the cone, then the slant height = 10 in., so that

$$\text{lateral surface} = 10\pi r = \text{area of the quadrant, } i.e. \ \tfrac{1}{4}\pi \cdot 10^2;$$

$$\therefore \ 10\pi r = 25\pi, \text{ from which } r = 2{\cdot}5 \text{ in.}$$

$$\text{Also slant height} = 10 = \sqrt{r^2 + h^2};$$

$$\therefore \ h^2 = 100 - 6{\cdot}25 = 93{\cdot}75,$$

or

$$h = 9{\cdot}68 \text{ in.}$$

Hence volume of cone $= \frac{1}{3}\pi \times 2{\cdot}5^2 \times 9{\cdot}68 =$ **63·4** cu. in.

104. The Trigonometry of the Cone. We have already seen from Art. 99 that the right circular cone is formed by the revolution of a right-angled triangle about one of the sides forming the right angle. Thus in Fig. 81, QOV is the generating triangle, and the angle OVQ opposite the base is clearly half the angle PVQ; hence it is called the **semivertical angle** of the cone. We shall denote the size of this angle as $a°$. The angle P′V′Q′ of the sector whose area gives the lateral area, we shall call the **sectorial angle** of the cone, and denote its size by $\theta°$.

PROBLEM 37. *Given the base radius r and the semivertical angle a, of a right circular cone, to find* (i) *its height,* (ii) *its lateral area,* (iii) *its sectorial angle, and* (iv) *its volume.*

(i) Referring to Fig. 81, $\angle$OVQ $= a$, OQ $= r$.

Let the height OV $= h$, then $h/r = \cot$ OVQ $= \cot a$;

$$\therefore \ \mathbf{h = r \cot} \, a.$$

(ii) VQ is the slant height, and $r/\text{VQ} = \sin a$,

so that

$$\text{VQ} = r \operatorname{cosec} a;$$

$$\therefore \ \textbf{lateral area} = \pi r \cdot \text{VQ} = \boldsymbol{\pi r^2 \operatorname{cosec}} \, a.$$

(iii) Let the sectorial angle P′V′Q′ $= \theta° = \theta\pi/180$ radians.

Now the circular measure of $\angle$P′V′Q′ = arc P′Q′/radius P′V′

$$= 2\pi r/l \text{ radians};$$

$$\therefore \ \frac{\theta\pi}{180} = \frac{2\pi r}{l} = 2\pi \sin a, \text{ so that } \theta° = \mathbf{360} \sin a.$$

(iv) Volume of cone $= \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \mathbf{r^3} \, . \, \mathbf{cot} \, a$.

Ex. 98. *A cup in the form of a right circular hollow cone has a capacity of* 231 *c.c. and its height is* 12·5 *cm. Find its semivertical angle and its sectorial angle, taking* $\pi = \frac{22}{7}$.

Let r cm. be the base radius, then

$$\tfrac{1}{3}\pi r^2 \times 12{\cdot}5 = 231,$$

from which $$r = 4{\cdot}2 \text{ cm.}$$

Now if α be the semivertical angle,

$$\tan\alpha = \frac{r}{h} = \frac{4{\cdot}2}{12{\cdot}5} = 0{\cdot}3360.$$

Hence, from the tables, $\alpha = \mathbf{18^\circ\ 34'}$.

Finally, if θ° be the sectorial angle,

$$\theta^\circ = 360 \sin 18^\circ\ 34' = 360 \times 0{\cdot}3184 = \mathbf{114^\circ\ 37'}.$$

105. Angles of a Pyramid. The angle between two intersecting planes is called a **dihedral** angle, and is measured by the plane angle between two straight lines, one in each plane, drawn from any point in the line of intersection, and perpendicular to it. The lines lie, therefore, in a third plane which is mutually perpendicular to the given planes. This definition enables us to calculate the dihedral angle between any two faces of a pyramid when sufficient measurements are known.

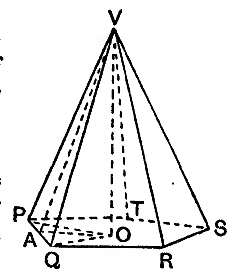


FIG. 80.

Ex. 99. *Find the angle between the base and* (i) *a triangular face,* (ii) *a slant edge of a regular pyramid of height* 19·3 *in. standing on a pentagonal base of side* 10 *in.*

Let VPQRST (Fig. 80) be the pyramid; since it is regular, the base PQRST will be a regular pentagon, and the perpendicular from V will meet it at O, the centre. Let OA be the perpendicular to PQ from O, then A is the mid-point of PQ, and AV is perpendicular to PQ.

Hence the angle between the base and the face VAQ is measured by the angle VAO.

Now, since all the sides of the pentagon are equal,

$$\therefore\ \angle \text{POQ} = \tfrac{1}{5} \,.\, 360^\circ = 72^\circ,$$

so that $$\angle \text{AOQ} = 36^\circ.$$

Hence $AO = AQ \cot 36° = 5 \times 1{\cdot}3764 = 6{\cdot}882$ in.;

$\therefore \tan VAO = OV/AO = 19{\cdot}3/6{\cdot}882 = 2{\cdot}805$;

$\therefore \angle VAO = \mathbf{70° \; 23'}$ approx.

The angle between a line and a plane is the angle between the line and its orthogonal projection upon the plane. For the edge VQ, the orthogonal projection on the base is QO, so that the angle between the edge and the base is VQO.

Now $QO = AO/\cos 36° = 6{\cdot}882/0{\cdot}809$;

$\therefore \tan VQO = OV/QO = (19{\cdot}3 \times 0{\cdot}809)/6{\cdot}882 = 2{\cdot}269$;

$\therefore \angle VQO = \mathbf{66° \; 13'}$ approx.

EXERCISES 15.

Take $\pi = 3{\cdot}14$, unless otherwise stated.

Find the total surface area of each of the following solids:

1. A cuboid measuring 8·5 in. by 7·5 in. by 11·5 in.

2. A right circular cylinder of length 7·6 ft. and diameter 4·8 ft.

3. A triangular prism 9 ft. 4 in. long, the sides of its triangular face measuring 7 ft. 5 in., 6 ft. 10 in., 4 ft. 9 in. respectively.

4. A triangular prism 21·6 cm. long whose right section is a triangle of sides 14·3 cm., 15·7 cm., 14 cm. respectively. Give the area in square inches, taking 6·45 sq. cm. to a sq. in.

5. A hexagonal prism 15·1 in. long, whose right section is a regular hexagon of side 8 in.

6. An octagonal prism 36 cm. long, whose right section is a regular octagon of side 5 cm.

7. A right pyramid 12 ft. high, standing on a rectangular base measuring 5·4 ft. by 4·4 ft.

8. A right pyramid 72 cm. high, standing on a rectangular base measuring 80 cm. by 42 cm.

9. A regular tetrahedron of side 5 in.

10. A right circular cone of height 24 cm. and base diameter 14 cm.

11. A right pyramid standing on a regular hexagonal base of side 3 in., whose slant edge is 11·3 in. long.

12. A right pyramid standing on a square base of side 77·4 cm., whose slant height is 176·3 cm. Give the area in square feet, using the equivalent of Ex. 4.

13. Find the whole surface of a solid right circular cylinder, three feet six inches high, and eighteen inches in diameter.

14. A square prism of 9·9 in. side and 18 in. long is cut by two parallel planes so that the ends become rectangles 10·1 in. by 9·9 in. Find its total surface area.

15. The cost of making a rectangular covered tank is 10d. per sq. ft. for the metal and 1*s*. per linear foot for riveting at the edges. What is the charge for making a tank 6 ft. by 3 ft. 9 in. by 2 ft. 8 in. if all the edges are riveted ? (D.S.)

16. Calculate the area of sheet lead required to line an open rectangular tank 11·6 ft. long and 8·4 ft. wide whose base slopes uniformly from a depth of 6 ft. to one of 7·3 ft. across the width of the tank.

17. An open cylindrical tank of diameter 25 ft. is to be constructed to hold 942 gallons of water. Taking $6\frac{1}{4}$ cub. ft. of water to a gallon, find the area of sheet iron required.

18. A rectangular lean-to shed has vertical walls and a rectangular sloping roof. The front wall is 9 ft. wide and 8 ft. high ; the back wall is 14 ft. high, and the distance between these two walls is 8 ft. Find the total area of walls and roof.

19. Find the lateral area of the wedge in Fig. 79 (p. 163), when PS = 75 ft., SR = 23·8 ft., ED = 51·2 ft., and the height of the ridge ED above the base PQRS = 12 ft.

20. A straight tunnel of semicircular section is 1 mile 25 chains long and 22 ft. wide on the ground. Find in square yards the total area of its internal curved surface. (J.M.B.)

21. The total surface area of a cube is 5046 sq. ft. Find the length of its edge.

22. The dimensions of a cuboid are as 3 : 4 : 5, and its total surface area is 2843·5 sq. ft. Find its dimensions.

23. From a sheet of iron 60·75 sq. ft. in area two hollow boxes are made, one being a cube and the other a cuboid whose dimensions are proportional to 10, 6, 3 respectively. If the total surface areas of the boxes are equal, find the dimensions of each.

24. The total surface area of a right circular cylinder is 12·56 sq. ft., and its length is 3·5 ft. Find the radius of its circular section.

25. A cylindrical boiler 19 ft. long has to be made from 1099 sq. ft. of iron plate. Find its diameter, allowing $\frac{1}{7}$th of area for overlapping and waste.

26. The total surface of a right circular cone whose slant height is 61 cm. is 2486·88 sq. cm. Find its base radius and its height.

27. The total area of tin-plate used in making a hollow cone, including the circular base of diameter 1 ft. 6 in., is 1413 sq. in. Find the height of the cone.

28. The area of the curved surface of a right circular cone is 2·6 times that of its base, and the height of the cone is 2 ft. 3 in. Find its total surface area in square feet.

29. The areas of the curved surfaces of a right circular cone and a cylinder of the same height and base diameter are as 65 : 126. Find the height of the cone when the base diameter is 1 ft. 4 in.

30. A right pyramid stands on a square base whose side is one inch less than three times the height of the pyramid, and the height is two inches less than the slant height of each triangular face. Find the total surface area of the pyramid in sq. ft.

31. The height of a right pyramid standing on a square base is h, and the length of one of its slanting edges is l. Shew that

(i) the area of its base is $2(l^2 - h^2)$, and

(ii) its lateral area is $2\sqrt{l^4 - h^4}$.

Hence calculate the total surface area of the pyramid when $l = 8{\cdot}5$ in. and $h = 8{\cdot}4$ in.

32. The curved surface of a right circular cone forms a semicircular area; find the semivertical angle of the cone.

33. A circular sector of radius 15·7 in. and angle 1·4 radians is made into a conical bag by bringing its straight edges together. Find the volume of the bag.

34. A hollow vessel made of very thin metal is of the form of a right circular cone. If it were slit along a generator and rolled out flat, it would become a quadrant of a circle of radius 20 in. What was the volume of the vessel?

35. A cone of height 18 in., and base diameter 12 in., is covered with thin leather over its curved surface. Find the sectorial angle of the leather. (N.U.T.)

36. A sector of a circle of radius 25 cm. has an angle of 135°. A hollow circular cone is constructed from this sector by bending it until its straight edges are just in contact. Find the volume of this cone. (N.U.T.)

37. ABCD is a regular tetrahedron, each of whose edges is 30 cm. long. Find (i) its height, (ii) its volume, (iii) the angle between the faces ABC, DBC, and (iv) the angle between the edge AC and the face BCD.

38. The angle between the base of a regular square pyramid and one of its triangular faces is 77° 12′. The side of the base is 2 ft. 4 in.; find the volume of the pyramid in cubic feet.

39. Find the volume and total surface of a right circular cone of height 25 cm., whose semivertical angle is 21° 48′.

40. The volume of a right circular cone of height 8 ft. 3 in. is 41,448 cub. in. Find the semivertical angle of the cone.

41. Find the angle made by the diagonal of a cube with each of the faces.

42. Find the angle between the diagonal of a rectangular prism measuring 82 cm. by 18 cm. by 39 cm. and the edge measuring 39 cm.

CHAPTER XVI

FRUSTA OF PYRAMIDS AND CONES. THE SPHERE

106. The Frustum. If a pyramid or cone be cut by a plane parallel to its base and the upper part containing the vertex be removed, the remaining solid is called a **frustum** of the pyramid or cone. Thus in Fig. 82 ABC*abc* is a frustum of a cone and PQRSTU*pqrstu* is a frustum of a pyramid.

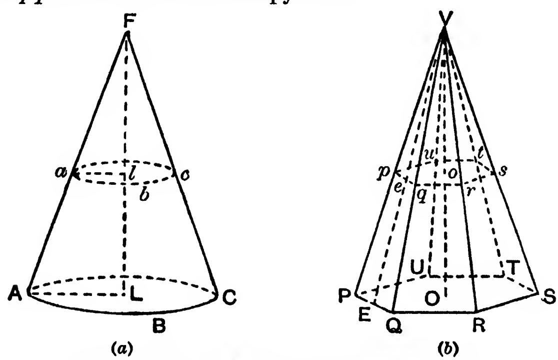


FIG. 82.—Frustum of a cone (*a*) and of a pyramid (*b*).

107. Volume of a Frustum. It will be easily understood that the volume of a frustum is the difference between the volume of the complete pyramid or cone and that of the smaller pyramid or cone which has been removed.

PROBLEM 38. *To find the volume of the frustum of a pyramid, and deduce the volume of a conical frustum.*

Let F be the area of the base PQRSTU (Fig. 82*b*) of the frustum of a pyramid, and f the area of the parallel face *pqrstu*.

Suppose in the complete pyramid the perpendicular from the vertex to the base meets the upper face in o and the base in O. Let $\mathsf{O}o=h$, then h is the perpendicular distance between the parallel faces.

Now from Prob. 29, $f/\mathsf{F}=o\mathsf{V}^2/\mathsf{OV}^2=(\mathsf{OV}-h)^2/\mathsf{OV}^2$.

Let us write k^2 for the ratio f/F, then

$$(\mathsf{OV}-h)/\mathsf{OV}=k,$$

from which

$$\mathsf{OV}=h/(1-k).$$

Hence, volume of frustum = vol. of $\mathsf{VP} \ldots \mathsf{U}$ − vol. of $\mathsf{V}p \ldots u$

$$\begin{aligned}
&=\tfrac{1}{3}\mathsf{F}\,.\,\mathsf{OV}-\tfrac{1}{3}f\,.\,o\mathsf{V}\\
&=\tfrac{1}{3}\mathsf{F}\,.\,\{\mathsf{OV}-k^2(\mathsf{OV}-h)\}\\
&=\tfrac{1}{3}\mathsf{F}\{(1-k^2)\mathsf{OV}+hk^2\}\\
&=\tfrac{1}{3}\mathsf{F}h(1+k+k^2)\\
&=\tfrac{1}{3}h(f+\sqrt{f\mathsf{F}}+\mathsf{F}),
\end{aligned}$$

so that the **volume of a frustum of a pyramid bounded by two parallel faces of areas f, F respectively and distant h apart is $\frac{1}{3}\mathbf{h}(\mathbf{f}+\sqrt{\mathbf{fF}}+\mathbf{F})$.**

In the case of a cone, if r, R be the radii of the parallel faces, $f=\pi r^2$ and $\mathsf{F}=\pi\mathsf{R}^2$, and the above expression becomes

$$\tfrac{1}{3}\pi h(r^2+r\mathsf{R}+\mathsf{R}^2),$$

i.e. the **volume of a frustum of a cone bounded by two parallel circular faces of radii r, R respectively, and distance h apart is $\frac{1}{3}\pi\mathbf{h}(\mathbf{r}^2+\mathbf{rR}+\mathbf{R}^2)$.**

Ex. 100. *A right pyramid* 21 *in. high stands on a triangular base whose sides are* 10·1 *in.*, 5·2 *in.*, 14·7 *in. respectively. It is cut by a plane* 12 *in. from the vertex and parallel to the base. Find the volume of the frustum thus formed.*

It is better in numerical examples to work from first principles rather than use the formula of Prob. 38.

Semi-perimeter of base $=\frac{1}{2}(10{\cdot}1+5{\cdot}2+14{\cdot}7)=15$ in.

$\therefore$ Area of base $=\sqrt{15\times4{\cdot}9\times9{\cdot}8\times0{\cdot}3}=14{\cdot}7$ sq. in.

Let f be the area of the parallel section, then

$$\frac{f}{14{\cdot}7}=\frac{12^2}{21^2}\text{ from which } f=4{\cdot}8\text{ sq. in.}$$

$$\begin{aligned}
\therefore\ \text{Volume of frustum}&=\tfrac{1}{3}\,.\,14{\cdot}7\times21-\tfrac{1}{3}\,.\,4{\cdot}8\times12\text{ sq. in.}\\
&=102{\cdot}9-19{\cdot}2=83{\cdot}7\text{ sq. in.}
\end{aligned}$$

Had we used the formula of Prob. 38, we should put $\mathsf{F}=14{\cdot}7$, $f=4{\cdot}8$, $h=9$, so that

$$\begin{aligned}
\text{Volume}&=\tfrac{1}{3}\,.\,9(4{\cdot}8+\sqrt{4{\cdot}8\times14{\cdot}7}+14{\cdot}7)=3(19{\cdot}5+8{\cdot}4)\\
&=83{\cdot}7\text{ sq. in.}
\end{aligned}$$

108. Lateral Area of a Frustum. We shall next find the measures of the lateral areas of the frusta of a pyramid and a cone respectively.

PROBLEM 39. *To find the lateral area of the frustum of a pyramid.*

Referring to Fig. 82*b*, it will be readily seen that each sloping face of a frustum of a pyramid is a trapezium; hence the slant height of each face must be found if the pyramid is not regular. If VeE be the slant height of the whole pyramid, then the area of the face $pq\mathrm{QP} = \frac{1}{2}(\mathrm{PQ} + pq) \times \mathrm{E}e$, and similarly for the other faces.

When the pyramid is regular, all the faces are equal in area, and the **lateral area of the frustum of a regular pyramid standing on a base with n equal sides is measured by the product**

$$\tfrac{1}{2}n(\text{sum of two parallel sides}) \times (\text{slant height})$$

or $$\tfrac{1}{2}(\text{sum of perimeters of parallel faces}) \times (\text{slant height}).$$

PROBLEM 40. *To find the lateral area of the frustum of a right circular cone.*

This might be deduced directly from Prob. 39, but an independent method will here be employed.

Let abcABC (Fig. 82*a*) be the frustum, R, r the radii of the parallel circular faces, ABC, abc. If the axis cuts abc in l and ABC in L, then AL = R, and $al = r$

Hence by Problem 36 (p. 167),

$$\begin{aligned}\text{Lateral area of frustum} &= \pi \mathrm{R} \,.\, \mathrm{AF} - \pi r \,.\, a\mathrm{F}\\ &= \pi\{\mathrm{R}(\mathrm{A}a + a\mathrm{F}) - r(\mathrm{AF} - \mathrm{A}a)\}\\ &= \pi\{(\mathrm{R} + r)\mathrm{A}a + \mathrm{R} \,.\, a\mathrm{F} - r \,.\, \mathrm{AF}\}.\end{aligned}$$

Now $\dfrac{al}{a\mathrm{F}} = \dfrac{\mathrm{FL}}{\mathrm{AF}}$, *i.e.* $\dfrac{r}{a\mathrm{F}} = \dfrac{\mathrm{R}}{\mathrm{AF}}$, so that $r \,.\, \mathrm{AF} = \mathrm{R} \,.\, a\mathrm{F}$;

$\therefore$ lateral area of frustum $= \pi(\mathrm{R} + r) \,.\, \mathrm{A}a$, or the

lateral area of a frustum of a right circular cone is measured by the product, $\pi \times$ (sum of radii of parallel faces) $\times$ (slant height).

We might write $\pi(\mathrm{R} + r) \,.\, \mathrm{A}a$ in the form $2\pi \,.\, \dfrac{\mathrm{R} + r}{2} \,.\, \mathrm{A}a$, and $\dfrac{\mathrm{R} + r}{2}$ is the radius of the section mid-way between the parallel faces; hence, the

lateral area of a conical frustum is measured also by the product, $2\pi \times$ (radius of mid-section parallel to base) $\times$ (slant height).

Ex. 101. *It is required to make a lamp shade in the form of a frustum of a right circular cone. The perpendicular height is to be* 7·5 *in., and the diameters of the circular ends* 18 *in. and* 10 *in. respectively. Find the minimum area of material required, allowing* 22 *sq. in. for overlap and waste.*

Let ABCD (Fig. 83) represent a vertical section of the complete shade through the axis of the frustum; then AB = 10 in., CD = 18 in., and AE, the perpendicular distance between AB and CD = 7·5 in.

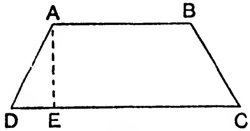


FIG. 83.

From Prob. 40, the lateral area

$$= \pi(5+9) \times AD$$
$$= \pi \times 14 \times \sqrt{DE^2 + EA^2}.$$

And $DE = \frac{1}{2}(DC - AB) = 4$ in.;

$\therefore$ lateral area $= \pi \times 14 \times \sqrt{16 + 56{\cdot}25} = 44 \times 72{\cdot}25 = 374$ sq. in.,

so that the total area of material required = (374 + 22) sq. in.

= **396** sq. in., or **2·75** sq. ft.

109. The Sphere. A solid whose surface is such that every point on it is equidistant from a fixed point within it is called a **sphere** (Fig. 84). The fixed point is called the **centre**, and the

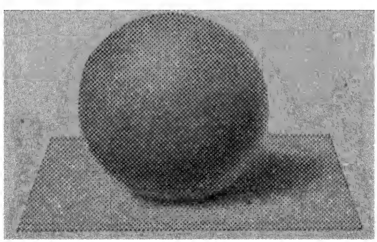


FIG. 84.

constant distance of every point on the surface from the centre is called the **radius** of the sphere. Any straight line terminated by the surface and passing through the centre is called a **diameter.** If a sphere is cut through at any part by a plane, the surface of the cut portion is a **plane section** of the sphere.

PROBLEM 41. *To find the shape of a plane section of a sphere.*

Let PBC (Fig. 85) be the section of a sphere cut by any plane KK′; draw, from the centre O of the sphere, ON perpendicular to

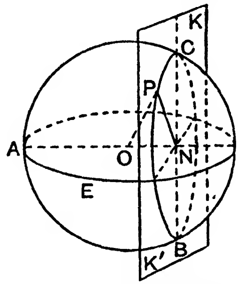


FIG. 85.—Plane section of a sphere.

the plane. Take any point P on the line of intersection of the plane and sphere, and join OP, PN.

Then since $\angle$ONP is a right angle, $NP^2 = OP^2 - ON^2$.

Now ON is constant in length for all positions of P, for it is the distance of the plane KK′ from the centre O. Likewise, since OP is the radius of the sphere, it is constant in length for all points on the line of intersection; hence NP is constant, so that the locus of P is a circle. Similarly, we may shew that any other plane section is a circle, so that

every plane section of a sphere is a circle.

From this fact it follows that a sphere is a **surface of revolution.** It is generated by the revolution of the semicircle ACA′ about its diameter AA′. If, therefore, two spheres intersect, the section is also a circle, for the straight line joining their centres is a common axis of revolution.

When a plane intersecting a sphere passes through the centre, the circle of section is called a **great circle.** All other circles of section are known as **small circles.** Thus in Fig. 85, AEA′ is a great circle, whilst CPB is a small circle.

110. Area of Spherical Zone. The portion of a sphere cut off between two parallel planes is called a **spherical frustum,** and the curved surface of such a frustum is known as a **spherical zone.** The constant distance between the parallel planes is called the **thickness** of the frustum. When the sphere is cut into two

portions by a single plane, either portion is called a **spherical segment**, and its curved surface is called a **spherical cap.**

PROBLEM 42. *To find the area of any spherical zone, and deduce the area of* (i) *a spherical cap, and* (ii) *the whole spherical surface.*

Let PQP′ (Fig. 86) be a semicircle whose centre is O. Suppose that in this semicircle half a regular polygon of an even number of sides be described, of which AB is one side. Let C be the mid-point of AB; join CO, and draw Aa, Cc, Bb each perpendicular to OP, and BE perpendicular to Aa.

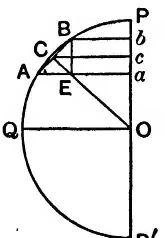


FIG. 86.—Area of a spherical zone.

Now, as the semicircle revolves about PP′ to generate a sphere, AB generates the frustum of a cone of which Cc is the radius of the mid-section. Hence by Prob. 40,

$$\text{Lateral surface of frustum} = 2\pi \,.\, \text{AB} \,.\, \text{C}c.$$

$$\text{But } \angle c\text{OC} = 90^\circ - \angle \text{OC}c = \angle c\text{CB} = \angle \text{EAB};$$

$$\therefore \sin c\text{OC} = \sin \text{EAB}, \quad i.e.\ \text{C}c/\text{OC} = \text{EB}/\text{AB} = ab/\text{AB},$$

$$\therefore \text{AB} \,.\, \text{C}c = \text{OC} \,.\, ab$$

so that lateral surface area becomes $2\pi \,.\, \text{OC} \,.\, ab$.

By making the sides of the polygon smaller and smaller, we can make the difference between AB and ab as small as we please, so that when the sides are increased in number indefinitely, the area of the zone traced out by the arc AB is $2\pi \,.\, \text{OC} \,.\, ab$,

i.e. **the area of a spherical zone is measured by the product**

2π (radius of sphere) × (thickness of zone)

Now for a spherical cap generated by the revolution about aP of the figure bounded by Aa, aP and the arc AP, the thickness becomes the distance aP, which is called the **height of the cap.** Hence, the **area of a spherical cap is measured by the product,**

2π (radius of sphere) × (height of cap).

If $a\text{P} = h$, and the radius of the sphere $= r$,

area of cap $= 2\pi rh$.

Finally, for the whole sphere the thickness becomes the diameter PP′, *i.e.* twice the radius, so that the **surface area of a sphere is measured by the product, 4π(radius of sphere)^2,** *i.e.* $4\pi r^2$.

If a cylinder be circumscribed about the sphere, its length would be 2π, and the radius of its circular base r; its lateral area would thus be $2\pi r \,.\, 2r = 4\pi r^2$. Hence the **surface area of a sphere is equal to the lateral area of its circumscribing cylinder.**

Ex. 102. *A bowl is to be made in the form of a spherical cap of height 9 in. whose top diameter is 2 ft. How many square feet of sheet metal will be required for it?*

Let ABC (Fig. 87) represent the elevation of the bowl, O, the centre of the sphere of which it forms part, and B, the mid-point

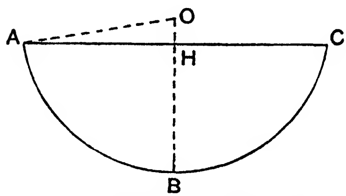


FIG. 87.—Surface of a spherical cap.

of the arc ABC. Then OB is perpendicular to AC, and bisects AC at H.

Now $AH = \frac{1}{2}AC = 1$ ft., and $BH = 9$ in. $= \frac{3}{4}$ ft.

To determine the radius OA of the sphere, we have

$$AO^2 = AH^2 + HO^2 = 1 + (OB - BH)^2.$$

Hence, if $AO = r = OB$, $r^2 = 1 + (r - \frac{3}{4})^2$, from which

$$r = \tfrac{25}{24} \text{ ft.};$$

$\therefore$ Area of metal required = area of cap

$$= 2\pi \,.\, \tfrac{25}{24} \,.\, \tfrac{3}{4} \text{ sq. ft.}$$

$$= \frac{3{\cdot}14 \times 25}{16} = 4{\cdot}9 \text{ sq. ft.}$$

111. Volume of a Sphere. We next pass to the determination of the volume of a sphere.

PROBLEM 43. *To find the volume of a sphere.*

Suppose the surface of the sphere to be divided into a large number of small triangles; then each triangle will be approximately plane, and may be regarded as the base of a pyramid whose vertex lies at the centre of the sphere. Thus in Fig. 88, DEBC is one of these pyramids, those around it having been removed.

Now, regarding each base as plane, we can, by making the triangles smaller and smaller, make their combined area differ as little as we please from the actual area of the spherical surface. Hence, ultimately the areas will be the same.

But the volume of a pyramid $=\frac{1}{3}$(area of base) $\times$ height, and the height of each pyramid = radius of sphere ; so that on summing the volumes of all the pyramids making up the sphere, we get :

$$\text{volume of sphere} = \tfrac{1}{3}(\text{surface area of sphere}) \times (\text{radius})$$
$$= \tfrac{4}{3}\pi(\text{radius})^3;$$

i.e. **the volume of a sphere is measured by the product, $\frac{4}{3}\pi(\text{radius})^3$.**

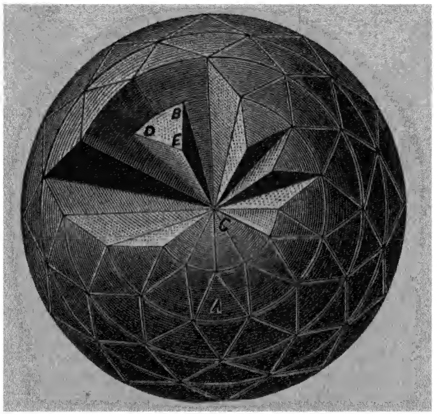


FIG. 88.—Sphere made up of small pyramids, a few of which have been removed.

It is usually more convenient to express the volume in terms of the diameter, since this is directly measured.

Let r, d be the radius and diameter of a sphere respectively, then the volume becomes $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi \frac{d^3}{8} = \frac{1}{6}\pi d^3$.

Putting $\pi = 3{\cdot}1416$, we have

$$\textbf{volume of sphere} = \tfrac{4}{3}\pi r^3 = \tfrac{1}{6}\pi d^3 = 0{\cdot}5236 d^3.$$

Ex. 103. *How many gallons of water will a hemispherical bowl 2 ft. 6 in. in diameter hold, taking* 0·16 *cu. ft. of water to the gallon ?*

Volume of hemisphere $= 0{\cdot}2618 \times 2{\cdot}5^3$ cu. ft.

$$= \frac{0{\cdot}2618 \times 2{\cdot}5^3}{0{\cdot}16} \text{ gallons}$$
$$= \mathbf{25{\cdot}56} \text{ gallons.}$$

112. Spherical Sectors and Segments. When a plane circular sector revolves about one of its bounding radii, it generates a

portion of a sphere called a **spherical sector.** Such a sector is shewn in Fig. 89, and it will be seen that the solid is made up of

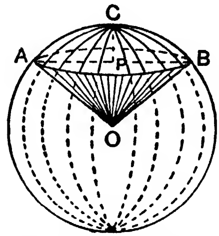


FIG. 89.—Spherical sector.

the spherical segment ABC and the right circular cone ABO, the vertex O being the centre of the sphere of which the sector forms part.

PROBLEM 44. *To find the volume of a spherical sector, and deduce the volume of* (i) *a spherical segment, and* (ii) *a spherical frustum.*

By dividing the spherical cap ABC into triangular elements, we may consider the sector made up of pyramids, just as in the case of the whole sphere.

Hence, by reasoning similar to that of Prob. 43, we have

the volume of a spherical sector is measured by the product,

$$\tfrac{1}{3}(\text{area of cap}) \times (\text{radius of sphere}).$$

If h = height of cap = PC, and r = radius of sphere, then

$$\textbf{volume of spherical sector} = \tfrac{2}{3}\pi r^2 h.$$

For the volume of the spherical segment, we have

$$\text{volume of ABC} = \text{vol. of sector AOBC} - \text{vol. of cone ABO}$$

$$= \tfrac{2}{3}\pi r^2 h - \tfrac{1}{3}\pi \text{AP}^2 . \text{OP}.$$

Now $\quad \text{OP} = \text{OC} - \text{PC} = r - h,$

and $\quad \text{AP}^2 = \text{OA}^2 - \text{OP}^2 = r^2 - (r-h)^2 = h(2r-h)$

so that the volume becomes

$$\tfrac{1}{3}\pi h\{2r^2 - (2r-h)(r-h)\} = \tfrac{1}{3}\pi h^2(3r-h).$$

To use this expression we must find r, the radius of the sphere, but it is usually more convenient to measure AP, the radius of

the base of the segment. Let AP $=a$, then it has just been shewn that AP$^2=h(2r-h)$, so that $r=\frac{1}{2}(a^2+h^2)/h$;

$$\therefore\ \tfrac{1}{3}\pi h^2(3r-h)=\tfrac{1}{3}\pi h\left(\frac{3a^2+3h^2}{2}-h^2\right)=\tfrac{1}{6}\pi h(3a^2+h^2).$$

Hence,

volume of spherical segment $=\frac{1}{3}\pi \mathbf{h}^2(3\mathbf{r}-\mathbf{h})=\frac{1}{6}\pi\mathbf{h}(3\mathbf{a}^2+\mathbf{h}^2)$,

where $h=$ height, and $a=$ radius of base of segment, and $r=$ radius of sphere of which the segment forms part.

For a frustum of a sphere, let its plane parallel surfaces be of distances h_1, h_2 from the surface of the sphere, and let h be its thickness, then taking $h_2>h_1$, $h=h_2-h_1$. The volume of the frustum is equal to the difference in the volumes of two spherical segments of heights h_1, h_2 respectively,

i.e. $\quad \frac{1}{3}\pi\{h_2{}^2(3r-h_2)-h_1{}^2(3r-h_1)\}$

$$=\tfrac{1}{3}\pi h\{3rh_2+3rh_1-(h_2{}^2+h_1h_2+h_1{}^2)\},\text{ since } h=h_2-h_1.$$

Let a, b be the radii of the upper and lower circular faces, then, as shewn above,

$$a^2=h_1(2r-h_1),\quad b^2=h_2(2r-h_2);$$

$\therefore$ By addition,

$$a^2+b^2=2r(h_1+h_2)-(h_1{}^2+h_2{}^2),$$

or

$$r(h_1+h_2)=\tfrac{1}{2}(a^2+b^2+h_1{}^2+h_2{}^2).$$

Substituting this value in the above expression, in order to get rid of r, we have

$$\tfrac{1}{6}\pi h(3a^2+3b^2+h_2{}^2-2h_1h_2+h_1{}^2)=\tfrac{1}{6}\pi h(3a^2+3b^2+h^2).$$

Hence, **the volume of a spherical frustrum of thickness h, and whose plane parallel faces have radii a, b respectively, is**

$$\tfrac{1}{6}\pi\mathbf{h}(3\mathbf{a}^2+3\mathbf{b}^2+\mathbf{h}^2).$$

Ex. 104. *A lead paper-weight in the shape of a spherical segment of height* 2·6 *cm. weighs* 5718 *grams. Find the radius of its spherical surface if lead weighs* 11·28 *gm. per c.c.*

Let r be the radius in cm. of the spherical surface, then

the volume of the solid $=\frac{1}{3}\pi(2{\cdot}6)^2(3r-2{\cdot}6)$ c.c.

$\therefore$ Weight of solid $=\frac{1}{3}\pi(2{\cdot}6)^2(3r-2{\cdot}6)\times 11{\cdot}28$ gm.

$$\therefore\ \tfrac{1}{3}\pi 2{\cdot}6^2(3r-2{\cdot}6)\times 11{\cdot}28=5718,$$

from which

$$3r-2{\cdot}6=\frac{5718\times 3}{\pi\times(2{\cdot}6)^2\times 11{\cdot}28}=71{\cdot}6\text{ cm., using logarithms;}$$

$$\therefore\ 3r=71{\cdot}6+2{\cdot}6=74{\cdot}2,\quad\text{and}\quad r=\mathbf{24{\cdot}7}\text{ cm.}$$

113. Lunes. When a sphere is cut by two planes each passing through the same diameter of the sphere, the portion ACBD (Fig. 90) of surface cut off is called a **lune**. It will be clear from Fig. 90 that a lune is bounded by the semi-circumferences of two

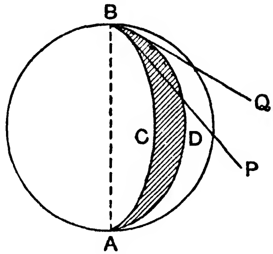


FIG. 90.—A lune.

great circles. The angle between the cutting planes is called the **angle of the lune**, and is measured by the angle PBQ contained between the tangents PB, QB to the great circles drawn in their respective planes.

Each of the solid portions of the sphere contained between the planes from the common diameter to the lunar surface is called a **spherical or lunar wedge**.

PROBLEM 45. *To find* (i) *the area of a lune and* (ii) *the volume of a spherical wedge.*

Let the angle of the lune be θ radians, its area, A, and the radius of the sphere, r. We may assume that two lunes of equal angle on the same sphere are equal in area.

Divide the given lune into n small lunes each of area a, where n is a large integer; then $A = na$, and the angle of each small lune is θ/n radians.

Hence the number of such lunes on the whole sphere $= 2\pi n/\theta$.

But this is also the number of times that the area a is contained in that of the whole spherical surface, *i.e.* $4\pi r^2/a$.

$$\therefore \frac{4\pi r^2}{a} = \frac{2\pi n}{\theta} \quad \text{or} \quad na = 2r^2\theta;$$

$$\therefore A = na = 2r^2\theta,$$

i.e. **the area of a spherical lune $= 2r^2\theta$, where θ is the angle of the lune in radians, and r is the radius of the sphere.**

For a spherical wedge whose curved surface is a lune of angle θ radians, let the lune be divided as before into n lunes of equal area, and suppose the volume of each corresponding wedge is v; then the volume V of the whole wedge $= nv$, assuming that wedges of equal angle in the same sphere are equal in volume.

The angle of each small wedge $= \theta/n$ radians, and the number of such wedges making up the whole sphere $= 2\pi n/\theta$.

But this is the number of times v is contained in $\frac{4}{3}\pi r^3$,

$$\therefore \frac{4\pi r^3}{3v} = \frac{2\pi n}{\theta} \quad \text{or} \quad nv = \tfrac{2}{3}r^3\theta;$$

$$\therefore \mathrm{V} = nv = \tfrac{2}{3}r^3\theta,$$

i.e. **the volume of a spherical wedge $= \frac{2}{3}r^3\theta$, where θ radians is the angle between its plane faces and r is the radius of its spherical surface.**

114. Spherical Triangles. A triangle formed on the surface of a sphere by the arcs of three intersecting great circles is called a **spherical triangle.** ABC (Fig. 91) is such a triangle. The angles

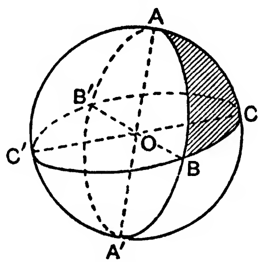


FIG. 91.—Spherical triangle.

of a spherical triangle are those between the planes containing the great circles of which the arcs are the sides.

PROBLEM 46. *To find the area of a spherical triangle.*

Let ABC (Fig. 91) be any spherical triangle, δ its area, and r the radius of the sphere. Suppose the measures of its angles be A, B, C radians respectively; then

$$\text{area of lune } \mathrm{ABA'C} = \delta + \Delta\mathrm{A'BC} = 2r^2\mathrm{A},$$

$$\text{,,} \quad \text{,,} \quad \mathrm{BAB'C} = \delta + \Delta\mathrm{B'CA} = 2r^2\mathrm{B},$$

$$\text{,,} \quad \text{,,} \quad \mathrm{CAC'B} = \delta + \Delta\mathrm{C'AB} = 2r^2\mathrm{C};$$

$\therefore$ By addition,

$$3\delta + \Delta\mathrm{A'BC} + \Delta\mathrm{B'CA} + \Delta\mathrm{C'AB} = 2r^2(\mathrm{A} + \mathrm{B} + \mathrm{C}).$$

Now $\triangle A'BC = \triangle AB'C'$, for they are corresponding portions of equal lunes;

$$\therefore \triangle A'BC + \triangle B'CA + \triangle C'AB = \triangle AB'C' + \triangle B'CA + \triangle C'AB$$
$$= \tfrac{1}{2}(\text{surface of whole sphere}) - \triangle ABC$$
$$= 2\pi r^2 - \delta;$$

so that
$$3\delta + 2\pi r^2 - \delta = 2r^2(A+B+C);$$
$$\therefore \delta = (A+B+C-\pi)r^2.$$

Since δ is always positive, it follows that

$$A+B+C > \pi \text{ radians or 2 right angles,}$$

i.e. **the sum of the three angles of a spherical triangle is greater than two right angles.**

The angle by which this sum exceeds π radians or two right angles is called the **spherical excess**, and is usually denoted by E; thus $A+B+C-\pi = E$, and the **area of a spherical triangle is measured by r^2E, where $E = A+B+C-\pi$, all the angles being expressed in radians.**

115. Latitude and Longitude. To locate a point P (Fig. 92) on the surface of a sphere, such as the Earth, two perpendicular

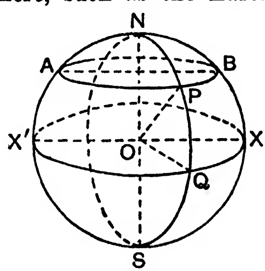


FIG. 92.—Latitude and longitude.

arcs NOS, X′OX are first chosen. The extremities N, S of the upright diameter SON are called **North and South Poles** respectively, and the great circle XQX′, the **equator.** For any point P on the surface, it is sufficient to know the angle POQ, made by the radius OP with the equatorial plane, and the angle XOQ which is made by the planes through NPQS and NXSX′. $\angle$POQ is usually denoted by θ, and is called the **latitude of P,** this being north or south according to whether P lies north or south of the equatorial plane.

The circle APB drawn through P parallel to the equatorial plane is known as a **parallel of latitude,** θ being constant for all points on the circle. The angle XOQ, usually denoted by ϕ, is called the **longitude of** P, this being east or west according to whether P lies in the eastern or western hemisphere. The great circle NPQS is called the **meridian of** P, and clearly ϕ will be constant for all points on this circle. Any point P, therefore, may be fixed on the surface if its latitude and longitude are known.

Ex. 105. *Regarding the Earth as a sphere whose mean diameter is* 7918 *miles, find how many miles Southampton, whose latitude is* 50° 54′, *is distant from the equator.*

Referring to Fig. 92, if P be the position of Southampton, then

$$\angle POQ = 50^\circ\ 54' = 0{\cdot}9570 \text{ radian.}$$

But the radian measure of $\angle POQ = \text{arc } PQ/OP = \text{arc } PQ/3959$;

$$\therefore \text{ arc } PQ/3959 = 0{\cdot}957, \text{ or arc } PQ = 0{\cdot}957 \times 3959$$
$$= 3789 \text{ miles.}$$

EXERCISES 16.

Take π as 3·14, unless otherwise stated.

1. Find the volume of a frustum of a right circular cone of height 6·5 ft. and radii of parallel faces 3·2 ft. and 1·5 ft. respectively. (U.L.C.I.)

2. Find the volume of the frustum of a cone whose circular faces are 11·3 in. and 5·1 in. in diameter respectively, the distance between these faces being 8 in.

3. A pail in the form of a frustum of a right circular cone is 19·3 in. deep, and the top and bottom internal diameters are 16 in. and 10·5 in. respectively. Find how many gallons of water it will hold, taking 6·25 cub. ft. to a gallon.

4. A frustum of a cone is generated by the revolution of a trapezium ABCD about BC, the parallel sides AB, CD each being perpendicular to BC. Calculate its volume when AB = 10 ft., CD = 6 ft. and BC = 8 ft.

5. ABCD is a trapezium in which

$$AB = 65 \text{ cm., } BC = 25 \text{ cm., } CD = 20 \text{ cm.,}$$

and the distance between BC and the parallel side AD = 16 cm. The figure revolves about AD; find the volume of the solid thus generated.

6. A tank has the shape of a hollow rectangular frustum. Its base is 8 ft. long and 6 ft. wide, and the top is 12 ft. long and 9 ft. wide. Its depth is 10 ft.; find its capacity in gallons taking 6·25 gallons to a cubic foot.

7. A symmetrical mound with sloping sides has a square base of 40 ft. side and a square top of 24 ft. side, and the distance between parallel edges is 10 ft. Find the number of cubic feet of earth in the mound. (H.S.)

8. A stone monument consists of a frustum of a square pyramid, with a complete square pyramid on the top whose base is equal to the upper face of the frustum. The height of the frustum is 8 ft., and the sides of its base and upper face are 3 ft. and 1·5 ft. respectively. The height of the complete pyramid is 2 ft. Find the volume of the monument.

9. A granite obelisk consists of the frustum of a square pyramid upon which a complete square pyramid rests, the base of this pyramid being equal to the upper face of the frustum. If the sides of the square faces of the frustum are 2 ft. 6 in. and 1 ft. 8 in. respectively; the height of the frustum 12 ft. 6 in., and the height of the complete pyramid is 1 ft., calculate the weight of the obelisk in tons, taking a cubic foot of granite to weigh 168 lb. (N.U.T.)

10. The volume of a frustum of a right circular cone is 836 cub. ft. Its base radius is 9 ft., and its height 6 ft. Find the radius of the upper face, taking $7\pi=22$.

11. The internal diameters of the top and bottom of a flower-pot are 10·5 in. and 7 in. respectively, and its depth is 7·5 in. Find the internal volume of the pot.

12. The height of a right circular cone is 20 ft. and the diameter of the base 24 ft.; the cone is cut by a plane parallel to the base at a perpendicular distance of 15 ft. from the vertex; find the area of the curved surface of the frustum. (H.S.)

13. How many sq. cm. of sheet iron would be required for a circular pail, without the handle, 29·7 cm. deep, whose top and bottom radii are 24 cm. and 18 cm. respectively.

14. The height of a frustum of a right cone is h, and the radii of its parallel circular faces are R, r; shew that the area of its curved surface is $\pi(\mathsf{R}+r)\sqrt{h^2+(\mathsf{R}-r)^2}$.

The dimensions of a frustum are $\mathsf{R}=11$ in., $r=2$ in., $h=40$ in.; find the radius of a circular cylinder, of the same height, which contains an equal volume. Prove that the total surface areas of the two solids are also equal. (H.S.)

15. A right pyramid 12 ft. high, standing on a rectangular base measuring 5·4 ft. by 4·4 ft., is cut by a plane parallel to the base and distant 6 ft. from it. Find the lateral area of the frustum.

16. Find the radius of the circle of intersection of two spheres of radii 5 in. and 7·3 in. respectively, their centres being 6·9 in. apart.

17. Two spheres intersect in a circle of radius 1 ft. 4 in. Their centres are 7 ft. 9 in. apart, and the radius of one of them is 5 ft. 5 in. Find the radius of the other.

18. A dome in the shape of an incomplete sphere rests on a circular base 10 ft. in diameter, and its height is 7 ft. Find its area. (H.S.)

19. Regarding the Earth as a sphere of diameter 7920 miles, find the area in square miles of that portion of its surface which should be visible from the top of a tower 264 ft. high.

20. Taking the radius of the Earth to be 3960 miles, prove that the area visible from a height of 100 ft. is rather more than 470 square miles. (H.S.)

21. Prove that the area of a spherical segment is the same as the area of a circle whose radius is the chord of half the arc of the segment. (H.S.)

22. Prove that the surface enclosed by the circle drawn with a pair of compasses is the same whether the circle is described on a plane or a sphere. (H.S.)

23. The diameter BA of a circle of radius r is produced to any point C, and a tangent CD is drawn touching the circle at D. DE is then drawn perpendicular to AB meeting it in E. Shew that the area of the spherical cap formed by the revolution about CE of the figure bounded by DE, EA and the arc AD is $2\pi r^2 h/(r+h)$, where CA $=h$.

If the area of this cap be 100·92 sq. in., and $r=14{\cdot}5$ in., find h and the base radius ED.

24. Find the volume of a sphere whose diameter is 2·15 ft.

25. Find the volume of a sphere whose diameter is 1·87 cm.

26. Find the radius of a sphere whose volume is $179\frac{2}{3}$ cu. ft., taking $\pi=\frac{22}{7}$.

27. Find the radius of a sphere whose volume is 1·987 c.c.

28. Taking the Earth and the Moon as spheres whose diameters are respectively 7926 miles and 2160 miles, how many times would a volume equal to that of the Moon be contained in a volume equal to that of the Earth? (L.S.)

29. A hemispherical basin holds 2·5 gallons of water. Find its diameter, having given that a cubic foot of water contains 6·25 gallons.

30. A boiler consists of a hollow cylinder 10 ft. long with hemispherical ends whose internal radii are $11\frac{1}{2}$ in. The thickness is $\frac{1}{2}$ in. throughout. Taking 500 lb. as the weight of a cubic foot of the iron of which the boiler is made, find the weight of the boiler. (H.S.)

31. A hemispherical bowl 15 in. in diameter is made from 2·5 sq. ft. of sheet iron. Find the area of the plate not used and the volume in gallons of the bowl, using the equivalent given in Ex. 29.

32. A solid sphere 6 in. in diameter weighs 5·4 lb., and it costs 15s. 9d. to plate its surface. What would be the weight of a sphere of the same material 10 in. in diameter, and how much would it cost to plate it? (D.U.)

33. A solid cube of lead, the length of whose edge is 12·9 in., is melted and cast into three solid spheres whose radii are in the ratio 1 : 2 : 3. Find the radii of these spheres assuming their total volume is exactly equal to that of the cube. (J.M.B.)

34. A solid cuboid of lead measuring 48·5 in. by 28·5 in. by 5·5 in. is melted down and recast into a hollow sphere. Assuming the shell to be of uniform thickness 5·25 in., find the internal and external diameters of the sphere. Take $\pi = \frac{22}{7}$.

35. A right circular cone 1 ft. 2 in. high has a base radius of 5·1 in.; find the radius of the inscribed sphere.

36. Into a hollow right circular cone standing on its vertex with its axis vertical, a sphere of radius 6·5 in. is placed. Find the volume of the space between the lower surface of the sphere and the vertex of the cone, the dimensions of the cone being: height = 2 ft., diameter of top = 1 ft. 8 in.

37. Assuming that the cone of greatest volume that can be inscribed in a given sphere is such that the distance of the centre of the sphere from its vertex is three-quarters of its altitude, shew that the greatest volume of a cone inscribed in a sphere of radius r is the volume of the sphere of radius $\frac{2}{3}r$. (H.S.)

38. A sphere of radius a is inscribed in a right circular cone of semi-vertical angle α and base radius b. Shew that the largest value of a is $b(\sec\alpha - \tan\alpha)$, and that the radius of the circle of contact with the curved surface of the cone is $b/(1 + \sin\alpha)$.

Compare the volumes of the sphere and the cone for the case in which $b = 9$, and $\tan\alpha = 0{\cdot}225$.

39. A sphere just fits into a cone whose vertical angle is 2α, and touches the base of the cone. If the radius of the base of the cone is a, prove that the radius of the sphere is $a\cos\alpha/(1 + \sin\alpha)$. (D.U.)

40. A sphere is inscribed in a cone so that a diameter of the circle of contact of the sphere and cone subtends an angle of 120° at the centre of the sphere; prove that the volume of the portion of the cone between the surface of the sphere and the vertex of the cone is one-eighth of the volume of the sphere. (H.S.)

41. The outside surface of a bowl is that of a frustum of a right circular cone, and the inside surface is a hemisphere of diameter 1 ft. The radii of the top and bottom of the frustum are 7 in. and 9 in. respectively, and its height is 7·5 in. Find the volume of the material of which the bowl is made.

42. A sector of a sphere is generated by the revolution of a circular sector whose angle is 60° about one of its bounding radii. Shew that the volume of its conical part is $\frac{3}{4}$ that of the whole section, and that the ratio of the area of the cap to that of the curved surface of the cone is as $2 : \sqrt{3}$.

43. How many gallons of water will a bowl hold whose shape is that of a spherical segment of depth 10 in. and upper diameter 2·5 ft., taking 6·25 gallons to a cubic foot ?

44. A sphere of radius 12·5 in. is cut by a plane distant 4·4 in. from the centre ; find the radius of the circle of intersection, and the volume in cubic feet of the smaller spherical segment.

45. A sphere of radius r is cut into two portions by a plane distant x from the centre. Find the volume of the smaller portion in terms of r and x.

If $3x = r$, find what fraction of the volume of the whole sphere the volume of the smaller portion is.

46. Find the volume of a frustum of a sphere of radius 5 ft. lying between two parallel planes on opposite sides of the centre at distances 3 ft. and 4 ft. respectively from the centre.

47. A sphere of radius 1 ft. 1 in. has a cylindrical hole of diameter 10 in. bored through it so that the axis of the hole is a diameter of the sphere. Find the volume in cubic feet of the sphere thus pierced.

48. A spherical cap is cut off by a plane whose latitude is $\theta°$; if be the radius of the sphere, shew that the surface of the cap is

$$2\pi r^2(1 - \sin \theta).$$

If this area be equal to that of a great circle of the sphere, find the value of θ.

49. Calculate in square miles the area of the Earth comprised between the 50th and 37th parallels of N. latitude and the meridians of longitude 13° W. and 40° E. Take the radius of the Earth as 3960 miles. (H.S.)

50. The volume of the frustum of a sphere of thickness 15 in. is 7065 cub. in. The radius of the upper face is 3 in. shorter than that of the lower face. Find the radii of these parallel faces.

MISCELLANEOUS EXERCISES

A.

1. How many rails 30 ft. long will be required for a mile of double track railway, and how many sleepers will be needed if there are 11 to each 30 ft. of single track ?

2. The speed of a ship is 23·1 knots ; how many miles per hour is this ?

3. The area of a rectangular field is 7·5 acres, and its length is 200 yards ; find its breadth. Find also the cost of enclosing it with a fence at 2s. 2d. per yard. (C.S.)

4. A rectangular vessel contains water to a depth of 65 mm. When 270 c.c. of water are added the depth increases to 80 mm. Find the area of the base of the vessel in square centimetres, and state in litres how much water it now holds. (O.S.)

5. At a distance of 125 ft. from the foot of a tower the angle of elevation of the top is 22° ; what is the height of the tower ?

B.

1. A map is drawn on a scale of 2 miles to an inch. What is the scale in kilometres to the centimetre ? Give the answer to three places of decimals. (L.S.)

2. Wall-paper is made in pieces each 12 yards long and 21 inches wide. In practice the number of such pieces required to cover the walls of a room is found by dividing the area of the walls in square feet by 54 to allow for waste in matching. What fraction of a piece is wasted ? How many pieces would be required to cover the walls of a room 13 ft. 6 in. long, 11 ft. 6 in. wide and 9 ft. high, allowing 4 per cent. of the total area for doors and windows ?

3. Find the number of litres of water which will fill a tank whose internal measurements are 3·2 metres long, 2·6 metres wide and 0·75 metre deep.

If it is a closed tank, find to the nearest centime the cost of painting it inside at 5 francs 30 centimes a square metre. (C.S.)

4. The sides BC, CA, AB of a triangle are 17 in., 10 in. and 21 in. in length respectively. Calculate the lengths of the segments into which AB is divided by the foot of the perpendicular dropped upon it from C, and thence find the length of this perpendicular. (L.S.)

5. ABC is a triangular grass plot right-angled at A, and a flag-pole 50 ft. high erected at A subtends angles of 10° and 5° at B and C respectively. Calculate the area of the plot in acres. (O.S.)

C.

1. Given that one inch is equivalent to 2·54 cm., find the number of chains in a kilometre.

2. A load of timber is 50 cub. ft., and boarded surfaces are measured by the "square," which is the area of a square of 10 ft. side. How many squares are there in a load of $1\frac{1}{4}$ in. thick floor boards, and how many loads of similar boards will be required to cover a rectangular floor 115 ft. long and 96 ft. wide?

3. Find the diameter of a circular running track in which there are four laps to the mile. Calculate also the area in acres of the plot enclosed by the track. Take $\pi = \frac{22}{7}$.

4. ABCD is a rectangle inscribed in a circle of radius 17·5 ft.; if AB is 21 ft. long, find the length of BC. If the rectangle represents a lawn, find the cost of turfing it at 3s. 9d. per square yard.

5. A ladder 32·5 feet long leans against a vertical wall, and the foot of the ladder is 8 ft. from the wall. How far from the ground is a man two-thirds of the way up the ladder, and what is the angle the ladder makes with the wall?

D.

1. If 1 metre = 39·37 in., how many metre-lengths can be cut from a ball of string containing 50 yards? Find to the nearest inch the length of the remaining piece. (O.J.)

2. The carpet 1 ft. 9 in. wide required to cover the floor of a room 36 ft. long by 29 ft. 9 in. wide cost £53 11s.; find the number of yards of carpet bought, and its price per yard. (L.S.)

3. Given that 1 cub. ft. = $6\frac{1}{4}$ gallons, and 1 cub. in. = 16·39 c.c., find the number of litres in a gallon to the nearest hundredth.

4. The external radius of a hollow metal pipe 10 metres long is 20 cm. The thickness of the metal is 1·5 cm. Find to the nearest kilogram the weight of the metal, taking one cubic centimetre of it to weigh 6·4 grams. Take π to be 3·14. (C.S.)

5. The front of a shed with a sloping roof is 8 ft. high; the back is $5\frac{1}{2}$ ft. high and the length of the sloping edge of the roof is 7 ft. What angle does the roof make with the vertical, and what is the width of the shed? (D.S.)

E.

1. A standard rod of brickwork is $272\frac{1}{4}$ square feet three half-bricks thick. Find the number of standard rods required for a wall 2994·75 ft. long, 11·1 ft. high and 2·5 bricks in thickness.

2. A series of soundings taken across a river channel is given by the following table, x ft. being the distance from one shore and y ft. the corresponding depth. Draw the section and find its area.

x	0	10	15	23	30	35	43	50	56	61	68	76	80
y	5	10	12	14	15	15	14	12	10	8	5	2	0

3. When the height of a cylinder is decreased by one-third, and its radius is reduced by 3 in., the volume of the cylinder is reduced by one-half. Find the radius of the cylinder. (O.S.)

4. In a triangle ABC, the angle B is 90°, the side CA is 62·5 ft. long, and the side AB is 49·25 ft. long. Find the length of the side BC and the size of the angle A.

5. If $\sin C = \sin A \cos B - \sin B \cos A$, find the value of the angle C when A is 65° and B is 32°.

F.

1. A small circle of 3·6 inch radius rolls around the outside of a larger circle of 16·4 inch radius once in 4 seconds. How many feet per minute does its centre travel ?

2. A triangular field ABC is to be divided into two portions of equal area by a straight fence from a point P in AB to a point Q in CA. If AB = 17 chains, BC = 10 chains, CA = 21 chains, and AQ = 14 chains, calculate the lengths of AP and AQ.

3. A rectangular block of iron of square section and 2·6 in thick weighs 65 lb. Find the side of the square section being given that the iron weighs 444 lb. per cubic foot. (L.M.)

4. A right circular cone 21 in. high whose base diameter is 12 in. is cut by a plane parallel to its base at a point 10 in. from the base. Find the volumes of the two parts.

5. The base BC of an isosceles triangle ABC is 5 in. long and the angle A is 36°. Calculate the length of the perpendicular from C to AB. (O.J.)

G.

1. A carpet 15 ft. long by 8 ft. 4 in. wide costs £11 19s. 7d.; what was the price per square yard ?

2. A boat brings 144 tons of coal averaging 42 cub. ft. to the ton. How long will it take to unload by a crane making 56 lifts per hour, each lift removing $20\frac{1}{4}$ cub. ft. ?

3. A right-angled triangle of sides 15 cm., 20 cm., and 25 cm. rotates about its hypotenuse as axis. Find the volume of the spindle thus generated. (C.P.)

4. ABC is a triangle and AD the perpendicular on BC. The angles B and C are 52·6° and 66·4° respectively, and BD is 523·1 ft. long.

Find (i) the length of D, (ii) the length of DC, and (iii) the area of ABC in acres.

5. AB and AD, adjacent sides of a parallelogram ABCD, are 10 ft. and 6 ft. long respectively, and contain an angle of 52°. Calculate the length of the diagonal AC, and its inclination to AB. (O.S.)

H.

1. The new Ordnance Survey plans are drawn to a scale of 1 to 500. Express this as inches to the mile, and give the true area of a field in acres which occupies 60·5 sq. in. on the plan.

2. The area of a square field is 20·8849 hectares ; find the length of one side, and express this length in yards, taking 1 yd. to 0·914 metre.

3. The conical roof of a turret 12 ft. in diameter is 11·25 ft. high. Find the slant height of the roof and the net area of sheet lead required to cover it. Find also the weight in cwt. of lead at 6·8 lb. per sq. ft.

4. The depth of an inverted hollow cone is 8 in. and its sloping surface makes an angle of 25° with its axis, which is vertical. If a ball of radius 1·5 in. is dropped into the cone, calculate the distance of the highest point of the ball below the mouth of the cone. (O.S.)

5. From a point in a plane through the foot of a tower the angle of elevation of the summit is observed to be 10°, and from a point 100 yards nearer it is observed to be 16°. Find the height of the tower. (L.M.)

I.

1. On a map the distance between two points 1210 yards apart is represented by 2·75 inches. Find the scale of the map in inches to the mile.

A point A is, according to the map, 3 inches East, and the point C 7·6 inches North of point B. Calculate the actual distances of A, B and C from one another to the hundredth of a mile. (C.W.B.)

2. Draw a triangle ABC having BC = 3·3 in., CA = 6·5 in., and angle B a right angle. In CB take a point P such that CP is 1·7 in. and produce BA to Q so that AQ is 0·7 in. Calculate the length of the line joining PQ, and find the difference in the areas of the triangles ABC, PBQ.

3. A field book record of a plot is as follows :

Links.

	770	
	660	330
	506	198
308	352	
	132	319
	0	

Find the area of the field in acres to two places of decimals.

4. The surface of the water in a rectangular tank 12 feet long and 9 feet wide stands at a level 6 feet higher than the surface of the water in another rectangular tank 10 feet long and 8 feet wide. Water is allowed to run out of the former tank into the latter until the surfaces are at the same level. How much will the surface of the latter tank rise, and how many gallons will have flowed from one tank to the other? Give the answers to the nearest tenth of an inch and tenth of a gallon respectively, taking $6\frac{1}{4}$ gallons to 1 cubic foot. (L.M.)

5. From the deck of a boat which is approaching a lighthouse the angle of elevation of the top of the lighthouse is 5°. One minute later it is 7° 30′. If the speed of the boat is 10 miles per hour, find the height of the lighthouse. (O.S.)

J.

1. A centimetre being 0·3937 inch, find to three significant figures how many acres there are in a square kilometre. (C.W.B.)

2. One of the equal straight edges of a fillet is 1 ft. 6 in. long. Find the area of the fillet. (See Ex. 21, p. 84.)

3. An embankment is 50 ft. high and its sides make angles of 60° with the horizontal. If the width of the embankment at the top is 30 ft., draw to a scale of 1 in. to 15 ft. a plan of the cross-section. Also calculate the volume in cubic yards of a portion of the embankment 150 yards in length. (O.J.)

4. An evaporating pan in the shape of a spherical cap 1 ft. 4 in. deep is made from a flat circular plate of iron 8 ft. in diameter; find the capacity of the pan in cubic feet.

5. At one corner B of a triangular field ABC stands a monument 120 ft. high. At A the elevation of the monument is 10°, and the side AC is found to make angles of 48° and 59° with AB and BC respectively. Find the lengths of the sides AB, BC of the field. (D.S.)

K.

1. The sides of a rectangular field of 25 acres are in the ratio 5 : 2. Find in yards the length of fencing to go round it. (C.S.)

2. ACB is an arc of a circle, C being the middle point of the arc. The length of the chord AB is 3·2 cm., and that of the chord AC is 2 cm.; find the length of the radius of the circle. (L.S.)

3. A cubic foot of silver weighs approximately 656 lb.; find the weight of a cubic centimetre in grams, having given that one cubic inch is equivalent to 16·4 c.c., and 1 lb. is equivalent to 453·6 grams.

4. A station is situated on a circular bend of a railway, and it is observed that the edge of the footboard of a coach 76 ft. long standing in the station has its ends vertically over the edge of the platform, whilst the middle point of the edge of the footboard is $9\frac{1}{2}$ in. horizontally from the platform. Assuming that the edge of the platform is circular, find the radius of the circle in yards.

5. Two angles of a triangle are 30° and 70°, and the side opposite the angle of 30° is 5 in. long. Calculate the lengths of the other two sides. (O.J.)

L.

1. Given that 1000 cubic centimetres are equivalent to 0·03532 cubic feet, and that one cubic centimetre of water weighs 0·002205 lb., find correctly to three significant figures the weight in lb. of one cubic inch of water. (C.W.B.)

2. Find the acreage of the following field:

	Links.	
	280	
	248	64
124	232	
	164	112
84	52	
	36	128
	0	

3. Calculate the area and the perimeter of a regular 10-sided figure inscribed in a circle whose radius is 3 inches. (O.S.)

4. If $\tan A = 48/55$, find, without using tables, the value of

$$(1 + \sec A)/(\sin A + \cos A).$$

5. Find the sides of the triangle ABC, being given that $A = 61° 14'$, $B = 43° 14'$, and that the length of the perpendicular from C on AB is 17·1 feet. (L.M.)

M.

1. A rectangular room 25 feet long by 15 feet wide is to be laid with carpet. An uncarpeted border is to be left one foot wide at each end and as narrow as possible along the sides of the room. The carpet is to be laid in strips parallel to the length of the room, each strip being 2 feet 9 inches wide. What will be the width of the border at each side of the room, and the length of the carpet required? What proportion of the area of the room will not be covered by carpet? (L.S.)

2. A series of soundings taken across a river is given in the following table, x being the distance in feet from one shore, and y the corresponding depth in feet. Draw the section and calculate its area in square feet:

x	0	10	25	35	40	58	65	70
y	0	5	8	10	15	12	7	0

3. A cylindrical vessel of thin tin, without a lid, is 6 in. high and the diameter of its base is 4 in. Calculate the number of square feet of sheet tin used in making 1200 such vessels. (O.J.)

4. A solid sphere 10·5 in. in diameter is placed in a rectangular cistern containing water to a depth of 19·3 in. The base of the cistern measures 49·5 in. by 17·5 in.; find how far the surface of the water is now from the base, taking $\pi = 3\frac{1}{7}$.

5. Find all the angles of the triangle whose sides are 45 ft., 395 ft., and 27 ft.

If an isosceles triangle were drawn with the 45 ft. side of the above triangle as base and on the same side of it, and having the same vertical angle, *i.e.* the angle opposite the 45 ft. side, as the above triangle, find the distance between the vertices of the two triangles. (L.M.)

N.

1. Find the cost of fencing a circular field whose area is 6 acres at a cost of 5s. per yard. (O.J.)

2. Find the number of areas in the following field, and show that the survey-line bisects the area:

Metres.

	798	168
	602	77
343	532	
	350	175
231	224	
	56	448
	0	

3. A large block of stone in the shape of a wedge ABCDEF has the following dimensions: the faces ABFE, CDEF meeting in the edge EF are trapeziums having AB = CD = 8 ft. 4 in., and EF = 4 ft. 4 in.; the ends ADE, BCF are isosceles triangles having AD = BC = 1 ft. 6 in., AE = ED = BF = CF = 3 ft. 5 in. Find the volume of the block in cubic feet, and calculate its weight, taking 88·5 lb. to a cubic foot.

4. A pyramid stands on a square base ABCD which is 4 in. by 4 in., and its vertex O is 5 in. above the base. The pyramid is divided into two parts by a plane section passing through the edge AB and the middle points of the edges OC, OD. Calculate the area of this plane section, and the angle between it and the base. (O.S.)

5. Two sides of a triangle are 13·45 ft. and 54·31 ft., and the included angle is 67° 24′; calculate the remaining angles. (L.M.)

O.

1. A sports ground has a circular track of the same width all round. The area of the ground including the track is 12 acres, and excluding the track, 10 acres. Find the width of the track, taking $\pi = \frac{22}{7}$. (L.M.)

2. Taking a cubic foot of water to weigh 1000 oz., what must be the diameter of the base of a cylindrical tank 8 feet deep in order that it may hold 39,250 gallons of water ?

3. ABCD is a square of side a; with centre A and radius AB, a circular arc BED is drawn and on BD as diameter a semicircle BCD is described. Calculate (i) the area of the figure BEDC bounded by the circular arcs, (ii) the area of the segment BDE, and prove that the segments on BC, CD are each equal in area to half that of the segment BDE.

4. Draw on a single diagram on squared paper the graph of sin A and 2 cos A for values of A between 55° and 70°, taking the side of one large square to represent 0·2 for numbers and 2° for angles. From the graph read off the value of sin A when tan A = 2. (O.J.)

5. In a triangle ABC, $a = 123{\cdot}4$, $b = 234{\cdot}5$, $C = 57°\ 28'$; find A and B. (C.W.B.)

P.

1. A solid wedge has one face, ABCD, a rectangle with AB 25 in., and BC 10 in. The ends FAD, EBC are equal isosceles triangles. The edge EF is 5 in. long, is parallel to the face ABCD, and is distant 8 in. from it. Find the volume of the wedge. (C.P.)

2. A lamp shade is to be made in the form of a hollow frustum of a right circular cone with the following dimensions: slant height 16 cm.; diameters of upper and lower circular sections, 3·6 cm. and 28 cm. respectively. Find the area of sheet metal required and the angle of the sectorial piece from which it must be cut.

3. Find the radius of a sphere which has the same volume as a cone whose height is 2 ft., and whose base is a circle 10 in. in diameter. (O.J.)

4. Find the radius of a circle in which a chord BC of length 27 cm. subtends an angle of 64° 30′ at the circumference. If another chord AB in the same circle is 18 cm. long, the angle BAC being acute, find the other angles and the other side of the triangle ABC as nearly as the tables permit. (L.S.)

5. ABC is a triangle right-angled at C, and the bisector of the angle A meets BC in D. Find the lengths of CD and DA; hence find the values of $\sin\frac{A}{2}$, $\cos\frac{A}{2}$, and $\tan\frac{A}{2}$.

Q.

1. ABCD is a square of side 8·5 inches; points E, F; G, H; K, L; M, N are marked on the sides AB, BC, CD, DA respectively such that the figure EFGHKLMN is a regular octagon. Calculate (i) the length of AE, and (ii) the area of the octagon.

2. A level field ABCD has three sides, AB, BC, CD, straight and of lengths 100 yd., 120 yd., and 140 yd. respectively, and the angles B and C are right. The side AD is curved, and if BC is divided into six equal parts, and at the points of division perpendiculars are drawn to BC to meet AD, their lengths in order are 110 yd., 115 yd., 130 yd., 135 yd. and 140 yd. Find approximately the acreage of the field. (C.P.)

3. A milk churn in the shape of a hollow conical frustum is 13 in. in diameter at the top, 20 in. in diameter at the bottom, and 3 ft. 4 in. high. Calculate the number of gallons it will hold, taking $6\frac{1}{4}$ gallons to a cubic foot.

4. A and B are two points outside a circular area. AB is 30 ft., and when produced passes through the centre. The angle between the tangents to the circle from B is 72°, and the corresponding angle at A is 52° 24′. Find the radius of the area to the nearest tenth of a foot. (C.W.B.)

5. PQ is a chord of a circle whose centre is O and radius 8 cm. A point R in PQ is such that OR is 5 cm. long and the angle PRO is 68° 17′; calculate the length of the chord PQ.

R.

1. Find the percentage of the area of a circle which lies within a regular octagon inscribed in the circle. (O.S.)

2. A cylindrical tank has a diameter of 4 feet. Find, to the nearest inch, the depth of water in the tank when the weight of the water is 1 ton. Take 1 cub. ft. of water to weigh $62\frac{1}{2}$ lb. (C.S.)

3. A granite monument consists of (i) a base of three square steps, each 10 inches high, the side of the step that rests on the ground being 7 feet long, and the sides of the other steps 5 feet and 3 feet, and (ii) a cylindrical column, 1 foot in diameter and 8 feet high, which stands upright on the highest step. If a cubic foot of granite weighs 169 lb., find to the nearest lb. the weight of the whole monument. (C.S.)

4. ACB is a circular arc whose centre is O; the chord AB is 19·8 in. long, and the chords AC, CB are each 10·1 in. long. Calculate the size of the angle AOB, and hence find the length of the arc.

Use this result to find the error in taking (8 . AC − AB)/3 as the length of the arc.

5. The sides b, c of a triangle are 478 and 563 yards respectively, and the angle A is 72° 12′. Find the length of the remaining side and the remaining angles, and calculate the area of the triangle. (L.S.)

S.

ABCD is a square described in a semicircle PDCQ, the points A, B lying in the diameter PQ. The figure represents a plot of ground; the square is covered to a depth of 6 in. with gravel costing 7s. 10d. per cubic yard, and the remainder of the semicircle is turfed at a cost of

12s. 8d. per 100 square feet. Find the total cost to the nearest penny if the radius of the semicircle is 24 feet. (L.S.)

2. The diameter of a cylindrical tobacco tin is 3 in., and its height is $4\frac{1}{2}$ in. Find the height of a cubical tin which will hold the same quantity of tobacco. (O.J.)

3. The height of a conical tent is 10 ft., and it is to enclose a circular area of 154 square feet. Calculate the area of canvas required for it.

4. ACB is the arc of a circle of radius 3 ft. standing on a chord AB, 3·6 ft. long; calculate the area of the segment ABC.

Calculate the area also from the expression

$$\tfrac{4}{3}h^2\sqrt{2r/h - 0{\cdot}608},$$

where h is the height of the arc and r the radius of the arc.

5. In a triangle ABC, a line AD is drawn meeting BC in D. Calculate the length of the segment BD when AB = 25 cm., BC = 52 cm., CA = 51 cm. and AD = 26 cm.

T.

1. A cubical tin standing on a base 3 in. by 8 in. contains water to a depth of 2 in. The tin is slowly tilted about one edge of the base until the water begins to overflow. Find the angle which the base then makes with the horizontal, and find through what further angle it must be tilted to get rid of half the water in the tin. (O.S.)

2. Find the area of a regular hexagon of side 3 in., and find how much it differs from the area of a circle which cuts off chords of 1 in. from each side of the hexagon. (L.B.)

3. A coal bunker is in the shape of a hollow frustum of a right square pyramid. Calculate the area of sheet metal required to construct the sides of one having the following dimensions: sides of parallel square sections, 12 ft. 10 in. and 2 ft. respectively; depth, 6 ft.

4. Three buoys A, B, C, are situated as follows: the distances AB, BC are each 200 ft., the bearings of A and of C from B are 25° S. of W. and 47° 30′ N. of E. Find, by calculation, the bearing of C from A and the length of AC. (J.M.B.)

5. From the top of a lighthouse 150 ft. high, two boats are observed simultaneously. One is seen in the direction 40° E. of N. at an angle of depression 10°, and the other in the direction 20° E. of S. at a depression of 15°. Calculate the distance between the two boats. (D.S.)

U.

1. In a quadrant of a circle of radius r, a circle is described touching the bounding radii CP, CQ of the quadrant at L, M respectively, and the arc of the quadrant at N. Prove that the radius of this circle is $(\sqrt{2}-1)r$, and calculate the area of the figure bounded by LP and the arcs LN, PN, when r is 15 cm.

2. A pyramid stands on a square base of area 400 sq. in., and has its vertex 30 in. above the centre of its base. Find its volume, the area of each of its triangular faces, and the lengths of its edges. (D.S.)

3. Two spheres of radii 13 in. and 15 in. have their centres 14 in. apart; prove that they intersect in a circle, and find its radius. If the two spheres are inscribed in the same cone, how far is its vertex from the centre of their circle of intersection? (O.S.)

4. The right section of the pier of a bridge is in the form of two equal circular segments, having the common chord as axis. The greatest thickness of the pier is 25 ft., and the common chord is 56 ft. long. Calculate the area of the section.

5. In a triangle ABC, A is 28°, a is 3·9 in., and b is 6·3 in. Find two possible values of B. (L.M.)

V.

1. A circular plate of iron of uniform thickness weighing 21·6 kilograms has two circular holes of 0·5 decimetre and 0·2 decimetre diameter drilled through it. If the weight is now 18·9 kilograms, find the diameter of the plate to the nearest millimetre. (C.W.B.)

2. On one side of a line AB, 10 cm. long, a semicircle ACB is described. On the other side, lines AF, BF are drawn each inclined at 45° to AB and meeting at F. With centre A and radius AB an arc is drawn cutting AF produced at P, and with centre B and radius BA an arc is drawn cutting BF produced at Q. With centre F and radius FP a quadrant PQ of a circle is drawn. Find the length of the oval curve ACBPQ, and the area enclosed by it. (L.S.)

3. Find the volume of the largest sphere which can be placed completely inside a hollow conical vessel 12 in. in height, whose base has a diameter of 10 in. (O.S.)

4. A door is 2 ft. 6 in. wide and 6 ft. high. Calculate the angle between the first and last position of a diagonal when the door is swung through 40°. (L.M.)

5. A triangle ABC has an area of 17·5 sq. in., and the angles A, B are 68° 12′ and 66° 48′ respectively. Find the lengths of its sides.

W.

1. Ten acres of land are to be divided among 120 allotments. Forty of these are to be equal, each having 10 sq. yd. more than each of the others. If a centimetre equals 0·3937 in., find (i) the number of square yards, (ii) the number of square metres, to the nearest integer, in the smaller allotments. (C.W.B.)

2. Two open cylindrical metal pipes have equal internal volumes. The internal and external diameters of one pipe are 12 in. and 13 in. respectively, and the corresponding diameters of the other are 6 in. and

$6\frac{3}{4}$ in. Express as a fraction in its lowest terms the ratio of the quantity of metal in the first pipe to the quantity in the second. (J.M.B.)

3. ABCD is a four-sided field; B is 250 links North and 300 links West of A, D is 450 links North and 200 links East of A, and C is 500 links North of A. The field is divided into two equal portions by a straight fence through B. Calculate the distances North and East of A of the point at which the fence meets AD. (O.S.)

4. Shew that the distance apart of two ships, when an observer on one ship who is 60 ft. above sea-level can just see the mast-head of the other ship, which is also 60 ft. above sea-level, is approximately 19 miles, assuming the earth to be a sphere of 3960 miles radius. (H.S.)

5. In a triangle ABC, a is 281 in., b is 445 in., and A is 18° 45′; find the difference between the two possible values of c. (L.S.)

X.

1. If a, b are the lengths of the parallel sides of a trapezium, and c is the length of the chord drawn parallel to these sides through the point of intersection of the diagonals, calculate c in terms of a, b. Also, if h is the height of the trapezium, find in terms of a, b, h, the areas of the two parts into which it is divided by this chord. (H.S.)

2. A piece of paper is bounded by two equal straight lines PA, PB each a inches long, and a circular arc ACB to which PA, PB are tangents. The shortest distance of P from the arc is b inches, and b is less than a. Shew that the radius of the arc is $\frac{1}{2}(a^2 - b)/b$ in., and find the area of the paper if a is 3 and b is 1. (L.M.)

3. OAB is a circular sector of radius r, whose angle AOB is θ; prove that the volume of the spherical sector generated by the revolution of OAB about OA is $\frac{2}{3}\pi r^3(1 - \cos\theta)$.

Calculate this volume when $r = 18$ in. and $\theta = 52°$.

4. AB is the diameter of a circle, and a chord AC when produced cuts the tangent at B in D. If AB is 10 in. and the angle BAC is 19° 48′, find the length of CD. (C.S.)

5. An equilateral triangle rests on a plane inclined to the horizon at an angle of 45°, one median of the triangle being along a line of greatest slope. Calculate the angle of inclination of the other medians to the horizontal. (O.S.)

Y.

1. ABCD is a trapezium in which AB is parallel to CD, and CB, DA produced meet at E. If AB is 13 cm., BC is 5 cm., CD is 18 cm., and DA is 6 cm., calculate (i) the size of the angle BEA, and (ii) the area of ABCD.

2. Find to the nearest gallon the amount of water a cylindrical tank with hemispherical ends holds if the external total length is 21 ft. 5 in.,

external diameter 3 ft. 7 in., and the thickness of the material $\frac{1}{2}$ in. Take $6\frac{1}{4}$ gallons to a cubic foot. (C.W.B.)

3. Find the area of a furnace flue whose right section consists of three sides BC, CD, DA of a rectangle, and a circular arc AEB having AB as its chord. Its dimensions are: BC = DA = 3 ft. 4 in., DC = AB = 3 ft. 8 in., height of arc AEB above AB = 1 ft.

4. ABC is a triangle right-angled at C. If BC is 12 cm. and CA is 11·9 cm., calculate the length of the bisector of the right angle intercepted between the hypotenuse and C.

5. The sides of a triangle are 3, 5, 7 units respectively. Find its greatest angle and the ratio into which the greatest side is divided by the perpendicular from the opposite angle. (L.M.)

Z.

1. Calculate the area in square miles which should be visible from the top of a hill 875 ft. high, assuming the earth to be a sphere of radius 3960 miles.

2. A right circular cone of height 2 ft. 9 in., and base radius 17·6 in., intercepts a sphere of radius 10 in. so that its axis passes through the centre of the sphere and its vertex is 1 ft. 9 in. from the centre. Find the radii of the two circles of section.

3. The roof ABCD of a shed is inclined at 32° to the horizontal, CD being the lower edge. The angle ADC is 65°, and each of the angles B, C is a right angle; find the inclination of AD to the horizontal.

4. ABCD is a rectangle; the lower edge AB and the upper edge DC are horizontal, and the plane ABCD is inclined at 40° to the horizontal. At C is a vertical rod CE, 6 ft. high. Find the angle EAC, if AB is 3 ft. and BC is 5 ft. (H.S.)

5. A, B, C and D are four points in the same vertical plane, of which A and B are on the ground at the same level and D is higher than C. It is desired to find the height of D above the level of AB, but D cannot be seen from B.

The following angles are measured: CAB = 130° 45′, ABC = 24° 20′, BCD = 28° 10′, BAD = 12° 50′. AB = 225 feet. Find the height of D above AB, by a trigonometrical calculation. (L.S.)

USEFUL DATA

$\pi = 3{\cdot}1416$, $\pi^2 = 9{\cdot}870$, $\dfrac{1}{\pi} = 0{\cdot}3183$, $\log \pi = 0{\cdot}4971$.

1 in. = 2·54 cm. ; 1 cm. = 0·3937 in.

1 ft. = 0·3048 m.

6 ft. = 1 fathom = 1·8288 m.

1 yd. = 0·914 m. ; 1 m. = 39·37 in. = 1·09 yd.

1 chain = 100 links = 22 yd. = 20·12 m.

1 geographical mile = 5280 ft. = 1·6 km. = 0·86 nautical mile.

5 geog. miles = 8 km.

1 nautical mile = 6080 ft. = 1·15 geog. mile.

1 knot = a speed of 1 nautical mile per hr.

1 sq. in. = 6·45 sq. cm. ; 1 sq. cm. = 0·155 sq. in.

1 sq. ft. = 9·29 sq. dm.

1 sq. yd. = 0·836 sq. m. 1 sq. m. = 1·19 sq. yd.

1 acre = 40·47 ares ; 1 are = 0·247 acre. 1 hectare = 2·47 acres.

1 cu. in. = 16·39 c.c. 1 c.c. = 0·061 cu. in.

1 cu. ft. = 28·32 litres. 1 c.m. = 35·32 cu. ft.

1 gallon = 0·1604 cu. ft. = 4·543 litres ; 1 litre = 1·76 pints = 6·103 cu. in.

1 lb. = 453·6 gm. ; 1 kg. = 2·204 lb.

1 radian = 57·3°. π radians = 180°.

1 cu. ft. of water weighs 62·3 lb.

TABLE OF AREA MULTIPLIERS FOR REGULAR POLYGONS

If R, r be the radii of the circum- and in-circles respectively of a regular polygon of n sides each of length a, then from p. 119, the area of polygon $=\frac{1}{4} na^2 \cot \frac{\pi}{n} = nR^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = nr^2 \tan \frac{\pi}{n}$.

Let $k_1 = \frac{1}{4} n \cot \frac{\pi}{n}$, $k_2 = n \sin \frac{\pi}{n} \cos \frac{\pi}{n}$, $k_3 = n \tan \frac{\pi}{n}$, then

$$\text{area of polygon} = k_1 a^2 = k_2 R^2 = k_3 r^2.$$

The following table gives the values of the multipliers k_1, k_2, k_3 for a number of polygons, including the equilateral triangle and the square :

n.	k_1.	k_2.	k_3.
3.	0·4333	1·2990	5·1963
4.	1·0000	2·0000	4·0000
5.	1·7205	2·3777	3·6325
6.	2·5981	2·5981	3·4641
7.	3·6335	2·7360	3·3714
8.	4·8284	2·8284	3·3136
9.	6·1818	2·8926	3·2760
10.	7·6942	2·9389	3·2492
11.	9·3646	2·9733	3·2307
12.	11·1962	3·0000	3·2154
15.	17·6423	3·0503	3·1890
16.	20·1128	3·0616	3·1824
18.	25·5208	3·0780	3·1734
20.	31·5690	3·0902	3·1680
24.	45·5748	3·1056	3·1608
25.	49·4738	3·1087	3·1575
30.	71·3580	3·1185	3·1530
36.	102·87	3·1248	3·1500
90.	644·4	3·1392	3·1429
180.	2578·05	3·1410	3·1419

Note how k_2 and k_3 approximate to π as n increases.

MENSURATION RULES

LENGTHS.

Prob.	Page.		
—	79.	Circumference of circle, radius r	$=2\pi r$.
10.	80.	Circular arc, radius r, $\angle\theta^\circ$	$=\theta\pi r/180$.
—	97.	,, ,, chord a, chord of half-arc, b (Approximate rule for small arcs)	$=\frac{1}{3}(8b-a)$.

AREAS.

1.	24.	Rectangle, sides l, b	$=lb$.
2.	28.	Parallelogram, base b, altitude h	$=bh$.
3.	29.	Triangle, base b, altitude h	$=\frac{1}{2}bh$.
6.	42.	,, sides a, b, c, perimeter $2s$	$=\sqrt{s(s-a)(s-b)(s-c)}$.
9.	71.	,, ,, a, b, included $\angle$C	$=\frac{1}{2}ab\sin \mathrm{C}$.
5.	34.	Trapezium, parallel sides a, b, altitude h	$=\frac{1}{2}h(a+b)$.
12.	87.	Circle, radius r, or diameter d	$=\pi r^2$ or $\frac{1}{4}\pi d^2$.
14.	94.	Circular sector, radius r, $\angle\theta$ radians	$=\frac{1}{2}r^2\theta$.
15.	96.	,, segment, ,, ,,	$=\frac{1}{2}r^2(\theta-\sin\theta)$.
—	98.	,, ,, ,, height h, chord a	$=\frac{4}{3}h^2\sqrt{2r/h-0{\cdot}608}$
—	,,	(Approximate rules)	or $\frac{1}{6}h(4a^2+3h^2)/a$.
42.	178.	Surface of sphere, radius r	$=4\pi r^2$.
,,	,,	Spherical zone, radius r, thickness h	$=2\pi rh$.
34.	165.	Curved surface of right cylinder, radius r, length l	$=2\pi rl$.
36.	167.	,, ,, ,, cone, radius r, height h	$=\pi r\sqrt{r^2+h^2}$.

VOLUMES.

23.	143.	Cuboid, edges l, b, h	$=bhl$.
25.	145.	Solid of uniform sectional area a, length l	$=al$.
31.	157.	Pyramid, base area a, height h	$=\frac{1}{3}ah$.
38.	173.	Frustum of pyramid bounded by parallel faces of areas f, F, and thickness h	$=\frac{1}{3}h(f+\sqrt{f\mathrm{F}}+\mathrm{F})$.
43.	179.	Sphere, radius r, diameter d	$=\frac{4}{3}\pi r^3$ or $0{\cdot}5236d^3$.
44.	181.	Spherical sector, height h, radius r	$=\frac{2}{3}\pi r^2h$.
,,	,,	,, segment, height h, base radius a	$=\frac{1}{6}\pi h(3a^2+h^2)$.
,,	,,	,, ,, ,, radius r	$=\frac{1}{3}\pi h^2(3r-h)$.
,,	,,	,, frustum, thickness h, radii of parallel faces a, b	$=\frac{1}{6}\pi h(3a^2+3b^2+h^2)$.

INDEX

The numbers refer to pages.

ANSWERS TO THE EXERCISES

Exercises 1. (Page 9.)

1. 385900·43.	2. 38128·85.	3. 9·87.	4. 1·96.
5. 1886·69.	6. 7·58.	7. 0·603.	8. 0·32.
9. 0·018.	10. 0·77.	11. 4·965.	12. 0·2478.
13. 19·42.	14. 1·319.	15. 32·83.	16. 5·619.
17. 0·051936.	18. 30·03.	19. 26·46.	20. 0·696.

21. $f=1{\cdot}85$. 22. 20·12 m., 6·75 chains.
23. 16·5 naut. m., 22·8 stat. m. 24. 471 chains.
25. 4892 km. 26. 3531 yards. 27. 0·505 %.
28. 23 pieces, 3·89 in. over. 30. 11 pieces, 1 m. 45 cm. over.
31. 72. 32. 12 pieces, $\frac{3}{8}$ in. over.
33. 17·5 inches, 0·655%. 34. 8882 m. 35. 0·612 in.
36. $15\frac{5}{8}$ mi. 37. 5 mi. 38. 8 min. 16 secs.
39. 0·15876 in. 40. 0·0000109 in.

Exercises 2a. (Page 14.)

1. 0·18.	2. 0·33.	3. 0·43.	4. 0·53.	5. 0·65.	6. 0·84.
7. 1.	8. 1·48.	9. 1·73.	10. 2·61.	11. 5·67.	12. 13.

Exercises 2b. (Page 19.)

2. PR = 2 in., OR = 3·75 in.,
tan OQS = tan OPR = 1·875 ; each angle = 61° 56′ approx.
4. tan POQ = 0·2917, ∠POQ = 16° 16′,
tan PQO = 3·4285, ∠PQO = 73° 44′.

5. 84° 54′.	6. 85° 36′.	7. 35°.	8. 36° 30′.
9. 19° 48′.	10. 50° 12′.	11. 21° 48′.	12. 68° 12′.

13. 23° 12′ + 21° 48′ = 45°. 14. 21° 48′ + 68° 12′ = 90°.
15. 84° 24′ − 16° 42′ = 67° 42′. 16. 0·3269, 0·3443, 0·7002 ; 53° 54′.
17. 0·1763, 0·3640, 0·8391 ; 70°. 18. 10·2, 3·1910, 13·3 ; 5° 12′.
19. 0·3, 2·3906, 9·5144 ; 84°. 20. 11·2, 0·4, 1·9711 ; 63° 6′.
21. $\theta = 76° 48' = A + B$. 22. $\theta = 52° 30' = A - B$.

23. $A=38°$, $B=52°$.
24. 222·4 ft.
25. 20·15 ft.
26. $BC=15$ in.
27. $CA=15$ in.
28. $CA=31·59$ cm.
29. $BC=65·94$ cm.
30. $CA=44·2$ in.
31. $BC=20$ ft.
32. $CA=4·085$ ft.
33. $BC=5·6$ in.
34. $B=C=62° 42'$, $A=54° 36'$.
35. $B=C=71° 12'$, $A=37° 36'$.
36. $BC=2$ ft. 6 in.
37. 17·2 in.
38. 70° 12′; 6·67 ft.
39. 76° 6′; 5·15 cm.
40. $B=61°$, $C=66°$.
41. $AD=20·37$, $C=71° 2'$.
42. $BD=40$, $DC=52·7$.
43. $AD=5$, $C=45°$.
44. $DC=27$, $B=151° 28'$.
45. $A=113° 12'$, $B=21° 48'$, $C=45°$.
46. 12° 12′.
47. $BC=20·3$ ft.
48. 32°, 5° 26′.
49. 402 ft.
50. Height of tower = 50·5 ft., length of pole = 13·6 ft.
51. 7 ft. approx.
52. 125 ft., 2° 11′.

Exercises 3a. (Page 31.)

1. $AB=3·7$ in. $AB^2=13·69$.
2. $AB=4·1$ in. $AB^2=16·81$.
3. $BC=6$ in. $AB^2=37·21$.
4. $AB=6·5$ in. $AB^2=42·25$.
5. $CA=3$ in. $AB^2=11·56$.
6. $CA=8$ cm. $AB^2=79·21$.
7. $CA=6·5$ cm. $AB^2=94·09$.
8. $CA=2$ cm. $AB^2=102·01$.
9. $BC=12$ cm. $AB^2=285·61$.
10. $AB=19·3$ cm. $AB^2=372·49$.

Exercises 3b. (Page 36.)

1. 2·25 acres.
2. 2·92 acres.
3. 9 sq. yd.
4. 25 sq. yd.
5. 60·4 acres.
6. 5 chains.
7. 19·8 ft.
8. 17 in.
9. 37·5 m.
10. £2 10s.
11. £49 7s. 5½d.
12. 5¼ sq. ft., 76 sq. in.
13. 460 ft.
14. 985 sq. ft., £1 0s. 6¼d.
15. 9 ft.
16. £19 10s.
17. 18.
18. £48 0s. 7d.
19. £29 15s.; £2 6s. 8d.
20. 4s. 6d.
21. 116·6 metres.
22. 183 sq. yd.
23. £26 6s. 6d.
24. $\frac{1}{7}$; 28.
25. 110 yards.
26. 219 yards.
27. 279 metres.
28. 1078 yards.
29. 741 inches.
30. 7 ft. 7 in.
31. 399 ft., 36 sq. yds., 139 × 127.
32. 10·125.
35. 16774·16 sq. cm.
36. 6·45.
37. 10 sq. yd.
38. 1512.
39. 1 ft. 5 in.
40. 2 ft. 11 in.
41. 9 ft. 11 in.
42. 17 in., 8·5 sq. ft.
43. 86 ft.
44. 48 ft.
45. 3·91 in.; 3·75 sq. in.
46. 16 yd. 2 ft.
47. 6·375 hectares.
48. 37·5 ft.
49. 6·2 mi.; 18·7 mi.
50. 12 in.; 126 sq. in.
51. 15 in.; 330 sq. in.
52. 40 in.; 1020 sq. in.
53. 72 ft.; 7884 sq. ft.
54. 84 ft.; 2016 sq. ft.
55. 88 yd.; 14916 sq. yd.
56. 63 yd.; 2394 sq. yd.
57. 21 cm.; 2520 sq. cm.
58. 15 ft.; 210 sq. ft.; 73·5 sq. ft.
59. 35 ft.

60. 33 yd. by 22 yd.; 2 yd.
61. 44 yd. by 11 yd.
62. 38 ft., 40 ft.
63. 630 sq. cm.
64. $x = 12$ ft.
65. 12 ft., 39·5 sq. yd.
66. 5·1 ft., 5·3 ft.
67. 6·62 in.
68. 40 chains.
69. 34·8 sq. chains; 2·1 chains.
70. 33 ft.; 65 ft.
71. 12 cm.; 13 cm.
72. 25 cm., 41 cm., 15 cm.
73. 804 sq. cm.

Exercises 4a. (Page 44.)

1. 84 sq. in.
2. 210 sq. in.
3. 1290 sq. ft.
4. 924 sq. ft.
5. 2280 sq. ft.
6. 95·7 sq. ch.
7. 1260 sq. ft.
8. 1092 sq. ft.
9. 136·08 sq. yd.
10. $6\sqrt{3} = 10{\cdot}393$ sq. yd.
11. $\frac{841}{4}\sqrt{3} = 364{\cdot}2$ sq. ch.
12. 454·4 sq. ch.
13. $4\sqrt{21} = 18{\cdot}33$ sq. in.
14. $18\sqrt{35} = 106{\cdot}5$ sq. in.
15. $12\sqrt{455} = 255{\cdot}97$ sq. in.
16. $36\sqrt{1218} = 1256$ sq. ft.
17. $42\sqrt{330} = 762{\cdot}97$ sq. ft.
18. 30·96 sq. ch.
19. 1226·85 sq. yd.
20. $168\sqrt{165} = 2158{\cdot}8$ sq. yd.
23. $\frac{1}{4}a\sqrt{3(a^2 - 4x^2)}$; $x = 1$ in.
24. 193·4 sq. in.
25. 4·66 sq. ft.

Exercises 4b. (Page 46.)

The following are the approximate areas of the figures.

1. 36·8 sq. ft.
2. 40 sq. m.
3. 2·78 sq. ch.
4. 1186·8 sq. yd.
5. 2102·5 sq. ft.
6. 7·08 sq. ch.

Exercises 4c. (Page 53.)

1. 264 sq. in.
2. 2589·6 sq. in.
3. 624 sq. ft.
4. 2856 sq. ft.
5. 20,196 sq. in.
6. 1932 sq. ft.
7. 1428 sq. ft.
8. 29·4 sq. ft.
9. 107 ac.
10. 1·15 ac.
11. 175 ac.
12. 5·45 ac.
13. 0·5 ac.
14. 1·78 ac.
15. 3·83 ac.
16. 4·61 ac.
17. 14·58 ac.
18. 7·24 ac.
19. 11·43 ac.
20. 51·06 ac.
21. 14·38 ac.
22. 18·91 ac.
23. 7·86 ac.
24. 13 ac.
25. 12 ac.
26. 4·84 ac.
27. 2·076 ac.
28. 5·32 sq. cm., 5·33 sq. cm.
29. 34·2 sq. in., 34·2 sq. in.
30. 63·93 sq. cm.; 63·95 sq. cm.
31. 63·37 sq. cm., 63·45 sq. cm.
32. 614·5 sq. cm., 614·6 sq. cm.

Exercises 5a. (Page 59.)

1. 0·208, 0·978.
2. 0·358, 0·934.
3. 0·423, 0·906.
4. 0·5, 0·866.
5. 0·588, 0·809.
6. 0·682, 0·731.

7. 0·743, 0·669. **8.** 0·839, 0·545. **9.** 0·887, 0·462.

10. 0·926, 0·378. **11.** 0·966, 0·259. **12.** 0·991, 0·131.

13. 0·7071, 0·7071. **14.** 0·894, 0·447. **15.** 0·96, 0·28.

16. 119/169 = 0·704, 120/169 = 0·710. **17.** 0·8, 0·6.

18. 0·996, 0·0906. **19.** 0·0698, 0·9976. **20.** 0·047, 0·9989.

21. $1/\sqrt{1+k^2}$, $k/\sqrt{1+k^2}$. **22.** $2mn/(m^2+n^2)$, $(m^2-n^2)/(m^2+n^2)$.

23. AD = 24, DB = 13·5, CD = 18 ; 0·8 ; 0·6.

Exercises 5b. (Page 67.)

1. 73° 44′ ; 0·28 ; 3·43. **2.** 53° 8′ ; 0·6 ; 1·33.

3. 30° ; 0·866 ; 0·577. **4.** 54° 6′ ; 0·586 ; 1·381.

5. 65° 30′ ; 0·415 ; 2·19. **6.** 65° 6′ ; 0·421 ; 2·15.

7. 27°. **8.** 32° 24′. **9.** 26° 45′ ; 0·893 ; 0·504. **10.** 45°.

11. 36° 52′ ; 0·6 ; 0·75. **12.** 69° 23′ ; 0·936 ; 2·66.

13. 64° 32′ ; 0·903 ; 2·10. **14.** 16° 16′ ; 0·28 ; 0·292.

15. 59° 20′ ; 0·86 ; 1·69. **16.** 74° 6′ ; 0·962 ; 3·51.

17. 37°, 30°, 23° ; 90°. **18.** 53°, 60°, 67° ; 180°.

19. A = 23° 36′ ; cos A = 0·9164 ; 47° 12′.

20. A = 25° 48′ ; sin A = 0·4352 ; B = 51° 36′.

21. Each value is approx. 2. **22.** 20°. **23.** 63° 12′.

24. 89° 12′ ; 0° 48′. **25.** A = 15° ; cos A = 0·9659.

26. A = 36° ; sin A = 0·5878 ; tan A = 0·7265.

27. $\frac{77}{85}$ = 0·9059 ; $\frac{36}{77}$ = 0·4676. **28.** $\frac{84}{85}$ = 0·9883 ; $\frac{84}{13}$ = 6·463.

29. $\frac{95}{193}$ = 0·4921 ; $\frac{168}{193}$ = 0·8704. **30.** 0·574 ; 0·819 ; 0·700 ; 1.

31. Each value = 1, approx. **33.** 22°, 32°, 48°, 64°, 82°.

34. 12°, 24°, 50°, 62°, 84°. **35.** 16°, 30°, 41°, 66°, 81° 24′.

36. b = 4·4, A = 69° 24′, B = 20° 36′. **37.** a = 2·4, A = 9° 36′, B = 80° 24′.

38. c = 37, A = 71° 4′, B = 18° 56′. **39.** c = 15·7, A = 57° 13′, B = 32° 47′.

40. c = 22·5, A = 73° 44′, B = 16° 16′. **41.** a = 24·04, b = 21·12, B = 41° 18′.

42. a = 52·37, b = 15·71, B = 16° 42′. **43.** b = 6·39, c = 30, B = 12° 18′.

44. b = 194·9, c = 200, B = 77°. **45.** a = 16·24, b = 49·4, A = 18° 12′.

46. a = 35·87, b = 77·28, A = 24° 54′. **47.** b = 58·46, c = 60, A = 13°.

48. b = 20·74, c = 50, A = 65° 30′. **49.** a = 239·7, c = 250, B = 16° 30′.

50. a = 4·876, c = 40, B = 83°. **51.** a = 15·436, c = 40, A = 22° 42′.

52. a = 56·35, c = 75, A = 48° 42′.

53. a = 117, b = 44, c = 125, A = 69° 23′, B = 20° 37′.

54. a = 15, b = 8, c = 17, A = 28° 4′, B = 61° 56′.

55. ∠CAD = 68° 43′.

56. AB = 3·42 cm., BC = 9·397 cm., ∠BAD = 75° 41′, AD = 13·88 cm.

57. 74° 13′. **58.** 18° 4′ ; 29° 28′.

59. 3·774 mi. ; 0·578 mi. **60.** 0·4472 ; 0·8944 ; 0·5 ; 53° 8′.

Exercises 6. (Page 75.)

1. $a=26$, $b=45{\cdot}03$.
2. $a=34{\cdot}05$, $b=66{\cdot}825$.
3. $b=200$, $c=282{\cdot}8$.
4. $c=420$, $a=346{\cdot}6$.
5. $b=12{\cdot}5$, $c=40$.
6. $c=250$, $a=239{\cdot}7$.
7. 451·44 ft.
8. 55·8 ft., 22·7 ft.
9. CA = 21·9 ch.; BC = 29·82 ch.
10. 266·1 ft.
11. (i) 32·3 ft., (ii) 28·1 ft.
12. 5·36 ft., 9·19 ft., 4·49 ft., 7·7 ft.
13. 16·2 in., 0·435 in.
14. 215 yd. 1 ft.
15. 86·9 yd.
16. 17·8 sq. yd.
17. 19·11 sq. yd.
18. 3795 acres.
19. 267·3 acres.
20. 2·49 acres.
21. 49·565 acres.
22. 23·82 hectares.
23. 6·67 hectares.
24. 12·95 acres.
25. 2·19 acres.
26. 140 sq. in.
27. 1117 sq. yd.
28. 56 acres.
29. 15·75 sq. m.
30. 3·58 ares.
31. 16·88 sq. ft.
32. 25·6 acres.
33. 82·8 sq. m.
34. 5·14 hectares.
35. 2·4 acres.
36. 72,600 sq. yd.
37. 2·25 sq. ft.
38. 42·42 sq. yd.
39. 43 yd., 48° 42′.
40. 1352 sq. ft.
41. 1491 sq. yd.
42. 2·15 in., 3·37 in., 3·99 in.
43. 1769·4 sq. yd.
44. 292·8 sq. in.
45. 736 sq. yd.
46. 16·24 sq. in.
47. 39·64 sq. in.
48. 25·46 hectares.
49. 402 ft.
50. 150·4 ft.
51. 108·5 sq. yd.
52. $a=50{\cdot}5$ cm.
53. $b=8$.
54. 1·93 acres.
55. 32·4.

Exercises 7a. (Page 82.)

1. 1·8 mi.
2. 4.
3. 1·04 km.
4. 55 in., 66 in., 99 in., 70 in.
5. 7·7 in.
6. 2·54.
7. 1·09.
8. 80 yd.
9. 19·25 in.
10. 2 ft. 1 in.
11. 2·5 mi. per hr.
12. 4320.
13. 2·54.
14. 62·5 mi.
15. 56 yd.
16. 15·75 in., 94·5 in.
17. 8·75 in., 7 in.
18. 43·75 mi. per hr.
19. 24 ft.
20. 4 ft. 8 in., 3 ft. 6 in.
21. 10·5 in.
22. 45 mi. per hr., 3·5 ft.
23. 54·16 cm.
24. 321·48.
25. 14·29 cm.
26. 448.
27. 7330 yd.
28. 1459·42 ft.
29. 83·4.
30. 155·6 yd.
31. 5·44 ft.
32. 26·5 ft.
33. 1·32 ft.
34. 2·2 m.
35. 13·2 yd.
36. 3·5°.
37. 57·27°.
38. 63° 2′.
39. 9 ft.
40. 1·09.
41. 6·5 cm.
42. 19·5 in.
43. 9·8 cm.
44. 19·8 cm., 28 cm.
45. 1760 yd.
46. 55 cm.
47. $d=(4h^2+l^2)/4h$, $h=3{\cdot}5$.
48. 24 in., 13 in.
49. 126 cm., 16 cm., 132 cm.
50. 45·2 ft.

Exercises 7b. (Page 87.)

The actual areas of the circles are given.

1. 1698·2 sq. cm. 2. 1905 sq. in. 3. 3217 sq. cm. 4. 3552 sq. cm.
5. 1210 sq. in. 6. 855·3 sq. in. 7. 30·68 sq. ft. 8. 1626 sq. ft.
9. 5058 sq. cm. 10. 3·1416 sq. yd. 11. 5476 sq. cm.
12. 7428 sq. cm.

Exercises 7c. (Page 90.)

1. 346·5 sq. in. 2. 13·86 sq. ft. 3. 962·5 sq. cm.
4. 260·26 sq. yd. 5. 38·5 sq. ch.
6. 498·96 sq. metres. 7. 9·62 sq. in. 8. 21·65 sq. cm.
9. 35·79 sq. ft. 10. 117·86 sq. in. 11. 907·92 sq. cm.
12. 5·73 sq. ch. 13. 7 ft. 14. 21 cm.
15. 154 yd. 16. 58·2 yd. 17. 52 cm.
18. 41·5 in. 19. 35·25 ft. 20. 60·5 yd.
21. 74·25 cm. 22. 98 m. 23. 11·2 in.
24. 9860 sq. yd. 25. 14 ft. 2 in. 26. 4·2 m. 27. 2·6 in.
28. 10·75 sq. ft. 29. (i) 0·48 sq. in.; (ii) 2·36 in.
30. 6 in. 31. £456 17s. 5d. 32. 10·5 ft.
33. 19·70 sq. in. 34. 7 ft. 35. $w=7$.
36. 8·56 sq. ft. 37. 13·2 sq. cm. The area is equal to that of the triangle.
39. 3·13.

Exercises 8. (Page 100.)

1. 3·1. 2. 2·46. 3. 0·85. 4. 4·536.
5. 2·125. 6. 2·05. 7. 20°. 8. 39°.
9. 47°. 10. 63° 36′. 11. 74° 36′. 12. 165° 42′.
13. 0·2496. 14. 0·6754. 15. 0·7854. 16. 0·9320.
17. 1·864. 18. 2·995. 19. 1 ft. 20. 1·44 chains.
21. 0·21 km. 22. 1·25 yd. 23. 37 ft. 24. 1280 yd.
25. 87° 40′, 866,800 mi., 9,441,000 mi. 26. (i) 40 cm., (ii) 21·7 cm.
27. (i) 3·74 in., (ii) 117° 24′ = 2·049 radians, (iii) 8·14 sq. in.
28. (i) 6·5 in., (ii) 119° = 2·0769 radians, (iii) 43·9 sq. in., 25·4 sq. in.
29. 3·014 mi. 30. (i) 17 in., (ii) 123° 54′ = 2·1625 radians, (iii) 36·8 in.
31. (i) 50° 6′ = 0·8744 radian, (ii) 8·57 in., (iii) 42 sq. in., 5·15 sq. in.
32. (i) 26·70 in., (ii) 26·67 in., (iii) −0·11 %.
33. (i) 20·33 in., (ii) 20·34 in., (iii) −0·05 %.
34. (i) 219·9 sq. in., (ii) 220 sq. in.
35. $c=\frac{1}{2}(20\pm\sqrt{157})=1{\cdot}87$ in., or 8·13 in. 36. 87·4 sq. in.
38. (i) 5·864, (ii) 0·0244, $r=900{\cdot}3$ ft. 39. 331 sq. in.
40. 1·1416 radians = 65° 25′; 57·08 sq. cm.
41. $r=4{\cdot}15$ in.; EF = 5·87 in. 42. 45; 45′. 43. (i) 6, (ii) 132·1 yd.

Exercises 9a. (Page 105.)

1. 60°.	2. 60°.	3. 45°.	4. 30°.
5. 44°.	6. 48°.	7. 39°.	8. 35°.

9. (i) AE = 4 cm., BF = 5·1 cm., CG = 7·5 cm., DH = 3·6 cm.
(ii) ∠XOG = 61° 57′, ∠X′OE = 28° 5′, ∠X′OF = 36° 53′, ∠HOX = 25° 4′.
(iii) ∠XOF = 143° 7′, reflex ∠XOE = 208° 5′, reflex ∠XOH = 334° 56′.

10. ∠ADC = 112° 18′, ∠AOC = 135° 24′, reflex ∠AOC = 224° 36′.

11. 12 in., 102° 40′.	12. 1·5 cm., 97° 38′.
13. 144° 42′, 125° 18′.	14. 154° 42′.
15. 192° 31′.	16. 314° 33′.

17. 252°.	18. 225°.	19. 200°.	20. 194° 24′.
21. 188°.	22. $180(n+2)/n$.	23. 36.	24. 12.
25. 9.	26. $4/(a-2)$.		

Exercises 9b. (Page 112.)

	Sine	Cosine	Tangent		Sine	Cosine	Tangent
1.	0·4695.	−0·8829.	−0·5317.	2.	0·6600.	−0·7513.	−0·8785.
3.	0·4289.	−0·9033.	−0·4748.	4.	−0·2756.	−0·9613.	0·2867.
5.	−0·5807.	−0·8141.	0·7133.	6.	−0·9781.	0·2079.	−4·7046.
7.	−0·7034.	0·7108.	−0·9896.	8.	−0·5090.	0·8607.	−0·5914.
9.	0·1908.	0·9816.	0·1944.	10.	0·7431.	−0·6691.	−1·1106.
11.	0	−1	0.	12.	−0·9877.	−0·1564.	6·3138.

13. 64·28, 57·36, −81·92, −76·60 ; 64° 9′ or 115° 51′.

22. 0.	23. −2·1196.	24. 25°.
25. 35°.	26. 30° 48′, 149° 12′.	27. 65° 30′, 114° 30′.
28. 47°, 313°.	29. 66° 18′, 293° 42′.	30. 21° 48′, 201° 48′.
31. 85° 36′, 265° 36′.	32. 197° 42′, 342° 18′.	33. 144° 6′, 215° 54′.
34. 135°, 315°.	35. 15°, 165°.	36. 36°, 324°.
37. 75°, 255°.	38. 165°, 195°.	39. 198°, 342°.
40. 108°, 288°.	41. ∞, 0, $-\infty$, 0.	43. 1·292.
44. 3·15.	45. 30°.	46. $\theta = 45°$ or 85° 36′.

Exercises 10a. (Page 115.)

In many of the following answers the values of the functions are given to four places, as in the tables, but it should be remembered that the result of a calculation is not accurate beyond the number of significant figures given.

1. 2.	2. 1·0683.	3. 1·2345.	4. 3·1352.
5. 1·1547.	6. 3·8637.	7. 1·3333.	8. 3·5714.
9. 1·0989.	10. 1·7700.	11. 1·4142.	12. 3·2361.

13. 2·5. **14.** 0·96. **15.** 0·0981. **16.** 0·0279.
17. 0·5774. **18.** 3·7321.

	Sin.	Cos.	Tan.	Cosec.	Sec.	Cot.
19.	0·1530.	0·9882.	*0·1548.*	6·5385.	1·0119.	6·4615.
20.	0·8988.	0·4382.	2·0513.	1·1125.	2·2821	*0·4875.*
21.	*0·9600.*	0·2801.	3·4269.	1·0417.	3·5698.	0·2918.
22.	0·8637.	*0·5040.*	1·7136.	1·1578.	1·9840.	0·5836.
23.	0·8660.	0·5000.	1·7321.	1·1547.	*2·0000.*	0·5774.
24.	0·6896.	0·7242.	0·9523.	*1·4500.*	1·3809.	1·0501.

31. $c=40$ in. **32.** $a=80$ cm. **33.** $b=12{\cdot}5$ ft.
34. $c=16$ yd. **35.** $\mathsf{A}=42°$, $b=7{\cdot}22$ in., $c=9{\cdot}71$ in.
36. $\mathsf{B}=38°$, $a=32$ ft., $c=40{\cdot}61$ ft. **37.** $\mathsf{B}=29°$, $b=17{\cdot}18$ in., $c=35{\cdot}45$ i
38. $\mathsf{A}=18°$, $a=13{\cdot}65$ ft., $c=44{\cdot}16$ ft. **39.** $\mathsf{A}=118°\ 36'$, $b=c=7{\cdot}44$ yd.
40. $\mathsf{B}=\mathsf{C}=68°\ 30'$, $b=c=47{\cdot}75$ ch. **41.** $\mathsf{C}=\mathsf{A}=71°$, $a=c=132{\cdot}1$ yd.
42. $\mathsf{B}=34°$, $a=c=123{\cdot}1$ cm. **43.** $\mathsf{C}=56°$, $a=b=100{\cdot}1$ yd.
44. $\mathsf{A}=\mathsf{B}=48°$, $a=b=83{\cdot}7$ cm.
45. $\mathsf{A}=50°$, $a=28{\cdot}78$ m., $b=28{\cdot}35$ m., $c=37{\cdot}1$ m.
46. $\mathsf{C}=19°\ 36'$, $a=16{\cdot}6$ m., $b=21{\cdot}2$ m., $c=7{\cdot}9$ m.

Exercises 10b. (Page 120.)

1. 150°, each of others = 125°.
2. $\angle\mathsf{BCD}=155°$, $\angle\mathsf{CDE}=70°$, $\angle\mathsf{DEF}=\angle\mathsf{EFA}=135°$.
3. 10. **4.** $\mathsf{A}=\mathsf{C}=\mathsf{E}=160°$, $\mathsf{B}=\mathsf{D}=\mathsf{F}=80°$, $\mathsf{AF}=1$ in., $\mathsf{AD}=1{\cdot}9$ in.
5. 118°, 108°, 62°, 72°. **6.** $\mathsf{A}=125°$, $\mathsf{B}=95°$, $\mathsf{C}=55°$, $\mathsf{D}=85°$.
7. 60°, 35°, 25°, 25°, 35°. **8.** $\mathsf{A}=\mathsf{B}=\mathsf{D}=108°$, $\mathsf{C}=54°$, $\mathsf{E}=162°$.
9. 128° 34′, 77° 8′, 51° 26′, 51° 26′, 141° 25′, 25° 44′, 12° 51′.
10. $\mathsf{B}=110°$, $\mathsf{C}=130°$, $\mathsf{D}=60°$, $\mathsf{E}=170°$.
11. $\mathsf{A}=50°$, $\mathsf{B}=80°$, $\mathsf{C}=130°$, $\mathsf{D}=100°$, $\mathsf{BD}=1{\cdot}32$ in.
12. $\angle\mathsf{RAD}=2\angle\mathsf{PRD}=162°$. **13.** 8.
15. $k_1=1{\cdot}7205$, $k_2=2{\cdot}3777$, $k_3=3{\cdot}6325$.
16. $k_1=2{\cdot}5981$, $k_2=2{\cdot}5981$, $k_3=3{\cdot}4641$.
17. $k_1=3{\cdot}6335$, $k_2=2{\cdot}7360$, $k_3=3{\cdot}3714$.
18. $k_1=4.8284$, $k_2=2{\cdot}8284$, $k_3=3{\cdot}3136$.
19. $k_1=7{\cdot}6942$, $k_2=2{\cdot}9389$, $k_3=3{\cdot}2492$.
20. $k_1=11{\cdot}1962$, $k_2=3$, $k_3=3{\cdot}2154$.
21. 555·8 sq. in. **22.** 204·5 sq. in. **23.** 522·4 sq. in.
24. 1063 sq. cm. **25.** 235·1 sq. in. **26.** 180·1 sq. cm.
27. 7·05 sq. ft. **28.** 1·8 ft., 2·82 sq. ft. **29.** 0·48 ac.
30. 23 cm. **31.** 2·8 in. **32.** $n=6$.
33. 56·92 sq. in. **34.** 2·22 in.

Exercises 11a. (Page 124.)

1. A = 41° 24′; B = 55° 46′; C = 82° 50′.
2. A = 78° 28′; B = 57° 7′; C = 44° 25′.
3. A = 90°; B = 67° 23′; C = 22° 37′.
4. A = 18° 55′; B = 90°; C = 71° 5′.
5. A = 75°; B = 60°; C = 45°.
6. A = 60°; B = 45°; C = 75°.
7. A = 63° 48′; B = C = 58° 6′.
8. A = 87° 30′; B = 67° 24′; C = 25° 6′.
9. A = B = 48° 54′; C = 82° 24′.
10. A = 67° 22′; B = 41° 7′; C = 71° 31′.

Exercises 11b. (Page 125.)

1. A = 31° 54′; B = 90°; C = 58° 6′.
2. A = 77° 19′; B = 56° 5′; C = 46° 36′.
3. A = B = 75° 45′; C = 28° 30′.
4. A = 90°; B = 47° 56′; C = 42° 4′.
5. A = 26° 17′; B = 112° 37′; C = 41° 6′.
6. A = 64°; B = 38° 41′; C = 77° 19′.
7. A = 17° 41′; B = 61° 56′; C = 100° 23′.
8. A = B = 72°; C = 36°.
9. A = C = 115° 19′; B = D = 64° 41′; AC = 95 yd. nearly.
10. A = C = 120° 55′; B = D = 59° 5′; BD = 86·54 m.

Exercises 11c. (Page 126.)

1. c = 7·8 in., A = B = 60°.
2. c = 8·5 ft., A = 90°, B = 57° 12′.
3. c = 37 cm., A = 37° 52′, B = 71° 4′.
4. a = 18·5 cm., B = 72° 3′, C = 35° 54′.
5. a = 16·8 yd., B = 29° 29′, C = 90°.
6. $a = 1 + \sqrt{3} = 2·73$, B = 45°, C = 60°.
7. b = 15 m., C = 130° 30′, A = 36° 52′.
8. a = 15·7 ch., B = 54°, C = 79°.
9. b = 533·4, A = 45° 22′, C = 62° 20′.
10. a = 245·5, B = 56° 53′, C = 50° 7′.

Exercises 11d. (Page 128.)

1. C = 58° 24′, a = 131 ft., b = 125 ft.
2. B = 90°, a = 21 yd., c = 20 yd.
3. C = 47° 33′, b = 20·5 m., c = 15·2 m.
4. A = 65°, b = 846·5 yd., c = 957·2 yd.
5. C = 27° 14′, a = 232·5 m., b = 160·3 m.
6. C = 90°, c = 100 in., a = 93·3 in.
7. B = 55° 54′, c = 365·6 cm., a = 328·5 cm.
8. C = 20° 54′, c = 12 ch., a = 31·38 ch.
9. C = 59° 30′, a = 12·3, b = 6·5.
10. A = 41° 54′, b = 469·5, c = 431·1.

Exercises 11e. (Page 130.)

1. C = 48° 36′, or 131° 24′; a = 102 in. or 33·2 in.
2. B = 90°, c = 4·4 ft.
3. No solution.
4. A = 58° 12′ or 121° 48′; b = 18·23 yd. or 0·733 yd.
5. A = 49° 7′, c = 64·94 m.
6. C = 54° 6′, a = 27·96 ch.
7. C = 90°, a = 24.
8. No solution.
9. B = 90°, c = 41·12.
10. B = C = 60°; c = 38·5.

Exercises 11f. (Page 130.)

1. A = 87° 4′, B = 38° 28′, C = 54° 28′.
2. A = 65° 48′, B = 35° 46′.
3. a = 255·5 yd., C = 60°.
4. A = 75° 59′, B = 29° 1′.
5. A = 90°, B = 61° 56′, C = 28° 4′.
6. C = 81° 6′ or 98° 54′; a = 502·9 m. or 352·6 m.
7. c = 698·8 ft.
8. BD = 11·2 cm., AC = 14·57 cm.
9. B = 72° 37′, C = 56° 3′.
10. A = 32° 12′.
11. C = 49° 12′ or 130° 48′.
12. 33° 42′, 56° 18′, 90°.
13. 23° 49′ or 156° 11′.
14. A = 80° 6′, b = 7·26 in., c = 11·31 in.
15. 139 sq. in.
16. $4c = 5b$, A = 82° 48′, B = 41° 24′, C = 55° 48′.
17. 93° 35′.
18. 55·47 yd.
19. 26 in., 30·81 in.; 81° 58′, 98° 2′.
20. BC = 8·84 in., CD = 7·12 in.
21. 2731 yards.
22. 6·96 in.
24. B = 65° 44′ or 114° 16′; C = 66° 16′ or 17° 44′.
25. AD = 40; ∠BAC = 6° 24′, ∠CAD = 30° 30′.
26. c = 25 or 19·8; area = 210 or 166·32.
27. A = 57° 55′; a = 4·3.
28. 8 in., 13·86 in.
29. 378 mi., 18° 22′ N. of E.
30. 7·1 mi., 5° 47′ N. of W.
31. 0·62.
32. AD = 23·52.
33. $b_1 : b_2$ = 125 : 147 = 0·85 : 1.
34. BC = 12 or 28.
35. a = 114·04; b = 144·23; tan C = 3.

Exercises 12. (Page 135.)

1. 33·7 ft., 66·2 ft.
2. 36° 12′ W. of N.
3. 625 yd.
4. 20 mi.
5. AP = 750·6 yd., AQ = 1965 yd.
6. 39·5 ft.
7. 931·7 yd.
8. 45° 49′ S. of E.
9. BD = 165·1 yd., BE = 198 yd.
10. 73·9 ft., 47·7 ft.
11. 1·8 mi.
12. 147·2 ft.
13. Ht. of each chimney = 122·5 ft., Distance apart = 257·8 ft.
14. 443·7 yd.
15. 1593 ft.
16. 1300 links.
17. 24·3 ft.
18. 21·2 ft.
19. 105·4 yd., 348 yd.

20. 4517·6 yd., 2 min. 58 sec. 21. 120 ft.
22. (i) 19,017 ft., 27,244 ft.; (ii) 19,266 ft. 23. 42·9 yd., 36° 52′.
24. AP = 498·2 yd., AQ = 609·89 yd., PQ = 315·07 yd.
25. 3·88 mi. 26. 276·2 yd. 27. 1·47 mi.
28. 11·03. 29. 62 ft. 30. 100 ft., 24° 40′.
31. 1st boat, 569·5 ft. N., 738 ft. E.; 2nd boat, 1310 ft. N., 552 ft. W. 2nd boat 740·5 ft. N., 1290 ft. W. of 1st boat.
32. 43·8 mi. per hr.; 490 ft. per min. 33. 135·6 ft.
34. 880 ft.

Exercises 13. (Page 151.)

1. 220·3 c. in. 2. 832 c.c. 3. 0·17 c. ft. 4. 3·86 c. ft.
5. 1·65 c. ft. 6. 63·5 c. ft. 7. 2·05 c. ft. 8. 0·24 stere.
9. 61 c. in. 10. 97 gallons. 11. 187·5 gal., 1875 lb. 12. 1·92 ft.
13. 1104 c. ft. 14. 0·325 m. 15. 9·9 in. 16. 0·309.
17. 7·58. 18. 621·7 c.c. 19. 12,154 tons. 20. 15 kg.
21. 7 in. 22. $\frac{1}{12}$ in. 23. 120·5 gallons. 24. 107 c. in.
25. 39,600. 26. 1,750,126 gallons. 27. 8·5 ft. 28. 17 ft. 4 in.
29. 5 ft. 30. 15 in. by 36 in. by 80 in. 31. 10 c. ft.
32. 18·4 c. ft., 39·27 sq. ft. 33. 7·5 c.c. 34. 6·4 c.c.
35. 1·5 litres. 36. 3·14. 37. 9406 tons. 38. 19·6 gal.
39. 0·77 %. 40. 2·75 cm. 41. 2 ft. 7 in. 42. 7·7.
43. 1800 gal., 1834 gal., −1·85 %. 44. 13 in. 45. 15·7 in.
46. 2 in. 47. 622·7 kg. 48. 23·6 tons. 49. 6·16 ft.
50. 0·704 in. 52. 4·15 sq. in., 2·3 in. 53. 2467·5 c.c.

Exercises 14. (Page 161.)

1. 53·76 sq. cm. 2. 12·56 sq. in. 3. 8·25 in. 4. 5·6 in.
5. 144 c. ft. 6. 56·7 c. ft. 7. 62 c. in. 8. 500 c. in.
9. 3·75 c. ft. 10. 9·1 c. ft. 11. 71·4 c.c. 12. 89·25 c. ft.
13. 0·68 c. ft. 14. 0·576 c. ft. 15. 554·24 c. in. 16. 1158·8 c. in.
17. 38·22 c. ft. 18. 6 ft. 19. 5 ft. 20. 448 lb.
21. 3 in. 22. 107·6 cm. 23. 113·7 c. in., 677·6 c. in.
24. 445·5 lb. 25. 17·32 in. 26. 17113·4 c.c. 27. 4·6 in.
28. 6 in. 29. 7·5 in., 3532·5 c. in. 30. 68 c. in.
31. 839·2 c.c. 32. 132·9 c. in. 33. 1·48 c. ft. 34. 3·14 c. ft.
35. 2355 c. in. 36. Vol. $= \frac{1}{6}bh(2l + e)$; 1 c. ft.

Exercises 15. (Page 170.)

1. 495·5 sq. in. 2. 150·72 sq. ft. 3. 209 sq. ft. 4. 176 sq. in.
5. 6·2 sq. ft. 6. 1681·4 sq. cm. 7. 143·76 sq. ft. 8. 12818 sq. cm.

9. 43·3 sq. in. 10. 703·36 sq. cm. 11. 124·2 sq. in.
12. 35·1 sq. ft. 13. 20 sq. ft. 14. 5·8 sq. ft.
15. £6 10s. 6d. 16. 364·6 sq. ft. 17. 3846·5 sq. ft. 18. 464 sq. ft.
19. 2535 sq. ft. 20. 15082·5 sq. yd. 21. 29 ft.
22. 16·5 ft. by 22 ft. by 27·5 ft.
23. Cube, 2·25 ft. side; cuboid, 3·75 ft. by 2·25 ft. by 1·125 ft.
24. 6 in. 25. 12 ft. 26. 11 cm., 60 cm.
27. 3 ft. 4 in. 28. 9·93 sq. ft. 29. 31·5 in. 30. 144 sq. ft.
31. 34·45 sq. in. 32. 30°. 33. 156 c. in. 34. 506·4 c. in.
35. 1·9857 rad. = 113° 46′. 36. 2134 c.c.
37. (i) 24·49 cm., (ii) 3181·5 c.c., (iii) 70° 32′, (iv) 54° 44′.
38. 9·32 c. ft. 39. 2617 c.c., 1160 sq. cm. 40. 11° 25′.
41. 35° 16′. 42. 65° 5′.

Exercises 16. (Page 186.)

1. 117·6 c. ft. 2. 442·4 c. in. 3. 0·25 gal. 4. 1642 c. ft.
5. 40192 c.c. 6. 4750 gallons. 7. 6272 c. ft. 8. 43·5 c. ft.
9. 4·2 tons. 10. 4 ft. 11. 456·7 c. in. 12. 384·6 sq. in.
13. 5013·3 sq. cm. 14. 7 in. 15. 90 sq. ft. 16. 4·8 in.
17. 2 ft. 10 in. 18. 232·5 sq. ft. 19. 1245 sq. mi.
23. $h = 1{\cdot}2$ in.; ED = 5·6 in. 24. 5·2 c. ft. 25. 3·42 c.c.
26. 3·5 ft. 27. 0·78 cm. 28. 49·38. 29. 13·8 in.
30. 1532 lb. 31. 6·75 sq. in.; 6·4 gal.
32. 25 lb., £2 3s. 9d. 33. 2·42 in., 4·84 in., 7·26 in.
34. 8 in., 13·25 in. 35. 3·57 in. 36. 282·3 c. in. 38. 225 : 1024.
41. 1063 c. in. 43. 14·675 gallons. 44. 11·7 in., 1·17 c. ft.
45. $\frac{1}{3}\pi(r-x)^2(2r+x)$; 7/27. 46. 454·2 c. ft. 47. 4·19 c. ft.
48. $\theta = 30°$. 49. 2,380,000 sq. mi. 50. 12 in., 9 in.

Miscellaneous Exercises. (Page 191.)

A. 1. 704, 3872. 2. 26·6. 3. 181·5 yd., £81 16s. 6d.
4. 180 sq. cm., 1·44 litres. 5. 50·5 ft.

B. 1. 1·267. 2. $\frac{1}{7}$ waste, 8 pieces. 3. 6240 litres, 134·3 francs.
4. 6 in., 15 in., 8 in. 5. 1·86 acres.

C. 1. 497·2. 2. 24, 23. 3. 140 yd., 3·18 ac.
4. 28 ft., £12 5s. 5. 21 ft., 14° 15′.

D. 1. 45, 28 in. 2. 204 yd., 5s. 3d. 3. 4·54 litres.
4. 1161. 5. 20° 56′, 6·54 ft.

E. 1. 203·5. 2. 811 sq. ft. approx. 3. 22·4 in.
4. 38·48 ft., A = 38°. 5. 33°.

F. **1.** 157 ft. per min. **2.** AP = 12·75 ch., PQ = 6·60 ch. **3.** 9·86 in.
4. 113·7 c. in., 677·6 c. in. **5.** 4·76 in.

G. **1.** 17s. 3d. **2.** 5 hr. 20 m. **3.** 3771·4 c.c.
4. (i) 684·2 ft., (ii) 298·9 ft., (iii) 6·46 ac. **5.** 14·5 ft., 19° 3′.

H. **1.** 126·7 in. to a mile; 2·41 acres. **2.** 457 m., 500 yd.
3. 12·75 ft., 240·3 sq. ft., 14·6 cwt. **4.** 2·95 in. **5.** 45·9 yd.

I. **1.** 4 in. to a mile; AB = 0·75 mi., BC = 1·9 mi., CA = 2·04 mi.
2. PQ = 6·5 in., 4·2 sq. in. **3.** 5·35 acres.
4. 3 ft. 5·4 in., 1723·4 gal. **5.** 229 ft.

J. **1.** 247·2 acres. **2.** 0·48 c. ft. **3.** 49050 c. yd.
4. 31·01 c. ft. **5.** AB = 785 ft., BC = 680·6 ft.

K. **1.** 1540 yd. **2.** $1\frac{2}{3}$ in. **3.** 10·5 gm.
4. 304·1 yd. **5.** 9·4 in., 9·85 in.

L. **1.** 0·0361 lb. **2.** 0·5 ac. **3.** 26·45 sq. in., 18·54 in.
4. 1·65. **5.** 24·97 ft., 19·51 ft., 27·58 ft.

M. **1.** $7\frac{1}{2}$ in., 115 ft., $\frac{47}{300}$. **2.** 614·5 sq. ft. **3.** 733·2 sq. ft.
4. 21 in. **5.** 83° 52′, 59° 30′, 36° 38′; 9·25 ft.

N. **1.** £151 2s. **2.** 3197·74 ac. **3.** 14 c. ft., 1239 lb.
4. 11·71 sq. in., 39° 48′. **5.** 98° 25′, 14° 11′.

O. **1.** 11·8 yd. **2.** 10 ft. **3.** (i) $\frac{1}{2}a^2$, (ii) $\frac{1}{4}(4-\pi)a^2$.
4. 0·894. **5.** A = 31° 45′, B = 90° 47′.

P. **1.** 733·3 c. in. **2.** 793·8 sq. cm., 274° 22′. **3.** 5·3 in.
4. 14·96 cm., B = 78° 30′, C = 37°, AC = 39·31 cm.
5. CD = 12·85 cm., DA = 49·01 cm.; 0·2747, 0·9624, 0·2857.

Q. **1.** (i) 2·49 in., (ii) 60·57 sq. in. **2.** 3·1 ac. **3.** 31·37 gal.
4. 53·2 ft. **5.** 12·8 cm.

R. **1.** 90·03%. **2.** 34 in. **3.** 12751 lb.
4. 45° 42′; 20·34 in., −0·01 in.
5. 617·2 yd., 60° 17′, 47° 31′, 128116 sq. yd.

S. **1.** £6 3s. 1d. **2.** 3·17 in. **3.** 268·3 sq. ft.
4. Both 211·8 sq. in. **5.** 17 cm.

T. **1.** 63° 27′, 12° 31′. **2.** 23·38 sq. in., 1·39 sq. in.
3. 239·8 sq. ft. **4.** 34° 15′ N. of E.; AC = 395·32 ft.
5. 1230 ft. very nearly.

U. **1.** 23·56 sq. cm.
2. Vol. = 4000 c. in., Area of face = 316·2 sq. in., Slanting edge = 33·17 in.
3. 12 in., 96 in. **4.** 969·6 sq. ft. **5.** 49° 19′, 130° 41′.

V. 1. 15·23 cm. 2. 36 cm., 99·55 sq. cm. 3. 155·2 c. in.
4. 15° 7′. 5. $a=7{\cdot}07$ in., $b=7$ in., $c=5{\cdot}39$ in.

W. 1. (i) 400 sq. yd., (ii) 334 sq. m. 2. 100/153.
3. 304 links N., 135 links E. 5. 483·7 in.

X. 1. $c=2ab/(a+b)$, $a^2(a+3b)\ h/\{2\ (a+b)^2\}$, $b^2(3a+b)\ h/\{2\ (a+b)^2\}$.
2. 1·7 sq. in. 3. 4693 c. in. 4. 1·22 in. 5. 20° 42′.

Y. 1. (i) 53° 8′, (ii) 74·4 sq. cm. 2. 1213 gal. 3. 13·2 sq. ft.
4. 8·33 cm. 5. 120°, 33 : 65.

Z. 1. 4121 sq. mi. 2. 8 in., 9·4 in. 3. 15° 19′.
4. 28° 43′. 5. 249·8 ft.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW.

Works on Practical Mathematics—*Continued*

A FIRST BOOK OF PRACTICAL MATHEMATICS. By T. S. USHERWOOD, B.Sc., and C. J. A. TRIMBLE, B.A. 2s. 6d.

PRACTICAL MATHEMATICS FOR TECHNICAL STUDENTS. By T. S. USHERWOOD, B.Sc., and C. J. A. TRIMBLE, B.A. Part I., 4s. Part II., 8s. 6d.

WORKS ON ALGEBRA AND GEOMETRY

A NEW ALGEBRA. By S. BARNARD, M.A., and J. M. CHILD, B.A. With or without Answers. Vol. I. Parts I.-III. 4s. 6d. Key, 7s. 6d. Vol. II. Parts IV.-VI. 5s. Key, 10s. Also in Parts.

EXERCISES FROM "A NEW ALGEBRA." Parts I.-IV. By S. BARNARD, M.A., and J. M. CHILD, B.A. 3s. 6d.

A SCHOOL ALGEBRA. By H. S. HALL, M.A. With or without Answers. Parts I.-III. 5s. 6d. Key, 12s. Also in Parts.

EXAMPLES IN ALGEBRA. Taken from Part I. of "A School Algebra." By H. S. HALL, M.A. With or without Answers. 2s. 6d. each.

ELEMENTS OF GEOMETRY. By S. BARNARD, M.A., and J. M. CHILD, B.A. Parts I.-VI. 5s. Also in Parts.

MACMILLAN AND CO., LTD., LONDON